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The Minimum Maximal Domination Energy of a Graph

Research Article

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Abstract: Small A dominating set D of a graph G is maximal if $V - D$ is not a dominating set of G. The maximal domination number $\gamma_m(G)$ of G is the minimum cardinality of a maximal dominating set in G. In this paper, we are introduced minimum maximal domination energy $E_D(G)$ of a graph G. We are computed minimum maximal domination energies of some standard graphs and a number of well-known families of graphs. Upper and lower bounds for $E_D(G)$ are established.

MSC: 05C50, 05C69.

Keywords: Minimum maximal dominating set, minimum maximal domination matrix, minimum maximal eigenvalues, minimum maximal energy of a graph.

1. Introduction

In this paper, we are conceder a simple graph $G = (V, E)$, that nonempty, finite, have no loops no multiple and directed edges. Let G be such a graph and let n and m be the number of its vertices and edges, respectively. The degree of a vertex v in a graph G, denoted by $d(v)$, is the number of vertices adjacent to v. For any vertex v of a graph G, the open neighborhood of v is the set $N(v) = \{u \in V : uv \in E(G)\}$. We refer the reader to [8] for more graph theoretical analogist. A subset D of vertices set V of G is called a dominating set of G if every vertex of $v \in (V - D)$ is adjacent to some vertex in D. The concept of maximal domination number was introduced by V. R. Kulli et al. [10]. A dominating set D of a graph G is maximal dominating set if $V - D$ is not a dominating set of G. The maximal domination number $\gamma_m(G)$ of G is the minimum cardinality of a maximal dominating set. Any maximal dominating set with minimum cardinality is called minimum maximal dominating (denoted MMD) set. For more details in domination theory of graphs we refer to [9].

The concept of energy of a graph was introduced by I. Gutman [6] in the year 1978. Let G be a graph with n vertices and m edges and let $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of A, assumed in non increasing order, are the eigenvalues of the graph G. As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G, i.e. $E(G) = \sum\limits_{i=1}^{n} |\lambda_i|$. For more details on the mathematical aspects of the theory of graph energy see [2, 7, 13]. The basic properties including various upper and lower bounds for energy of a graph have been established in [12, 14], and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [4, 5]. In this paper, we are MMD energy of a graph $E_D(G)$. We

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are computed MMD energies of some standard graphs and a number of well-known families of graphs. Upper and lower bounds for $E_D(G)$ are established. It is possible that the MMD energy that we are considering in this paper may be have some applications in chemistry as well as in other areas.

2. The minimum maximal domination energy

Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G)$. A dominating set D of a graph G is maximal dominating set if $V - D$ is not a dominating set of G. The maximal domination number $\gamma_m(G)$ of G is the minimum cardinality of a maximal dominating set in G. Any maximal dominating set with minimum cardinality is called a MMD set. Let D be a MMD set of a graph G. The MMD matrix of G is the $n \times n$ matrix $A_D(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_iv_j \in E; \\ 1, & \text{if } i = j \text{ and } v_i \in M; \\ 0, & \text{othewise.} \end{cases}$$

The characteristic polynomial of $A_D(G)$ is denoted by

$$f_n(G, \lambda) := det\left(\lambda I - A_D(G)\right).$$

The MMD eigenvalues of the graph G are the eigenvalues of $A_D(G)$. Since $A_D(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The MMD energy of G is defined as:

$$E_D(G) = \sum_{i=1}^{n} |\lambda_i|$$

We first compute the MMD energy of a graph in Figure 1.

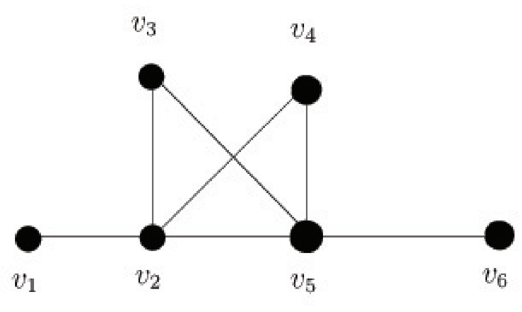

Figure 1.

Example 2.1. *Let G be a graph in Fig. 1 with vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and let its MMD set be $D_1 = \{v_1, v_2, v_5\}$. Then*

$$A_{D_1}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{D_1}(G)$ is

$$f_n(G,\lambda) = \lambda^6 - 3\lambda^5 - 4\lambda^4 + 6\lambda^3 + 5\lambda^2 - \lambda.$$

Hende, the MMD eigenvalues are $\lambda_1 \approx 0.3433,\ \lambda_2 \approx -1.4142,\ \lambda_3 \approx 1.4142,\ \lambda_4 \approx -0.8342,\ \lambda_5 \approx 3.4909,\ \lambda_6 \approx 0.0000.$ *Therefore the MMD energy of G is*

$$E_{D_1}(G) \approx 7.4468.$$

If we take another MMD set of G, namely $D_2 = \{v_2, v_3, v_5\}$, then

$$A_{D_2}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{D_2}(G)$ is

$$f_n(G,\lambda) = \lambda^6 - 3\lambda^5 - 4\lambda^4 + 6\lambda^3 + 3\lambda^2 - 3\lambda,$$

the MMD eigenvalues are $\lambda_1 \approx -1.2618,\ \lambda_2 \approx 0.6601,\ \lambda_3 \approx 3.6017,\ \lambda_4 \approx -1.0,\ \lambda_5 \approx 1.0000,\ \lambda_6 \approx 0.0000.$ *Therefore the MMD energy of G is*

$$E_{D_2}(G) \approx 6.9236.$$

This example illustrates the fact that the MMD energy of a graph G depends on the choice of the MMD set. i. e. the MMD energy is not a graph invariant. In the following section, we introduce some properties of characteristic polynomials of MMD matrix of a graph G.

Theorem 2.2. *Let G be a graph of order and size n, m. Let*

$$f_n(G,\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + ... + c_n$$

be the characteristic polynomials of MMD matrix of a graph G. Then

1. $c_0 = 1$.
2. $c_1 = -|D|$
3. $c_2 = \begin{pmatrix} |D| \\ 2 \end{pmatrix} - m$

Proof.

1. From the definition of $f_n(G,\lambda)$.
2. Since the sum of diagonal elements of $A_D(G)$ is equal to $|D|$, where D is a MMD set of a graph G. The sum of determinants of all 1×1 principal submatrices of $A_D(G)$ is the trace of $A_D(G)$, which evidently is equal to $|D|$. Thus, $(-1)^1 c_1 = |D|$.

3. $(-1)^2 c_2$ is equal to the sum of determinants of all 2×2 principal submatrices of $A_D(G)$, that is

$$\begin{aligned} c_2 &= \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1 \le i < j \le n} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= \sum_{1 \le i < j \le n} a_{ii}a_{jj} - \sum_{1 \le i < j \le n} a_{ij}^2 \\ &= \begin{pmatrix} |D| \\ 2 \end{pmatrix} - m. \end{aligned}$$

□

Theorem 2.3. *Let G be a graph of order n. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of $A_D(G)$. Then*

(i) $\sum_{i}^{n} \lambda_i = |D|$.

(ii) $\sum_{i}^{n} \lambda_i^2 = |D| + 2m$.

Proof.

(i) Since the sum of the eigenvalues of $A_D(G)$ is the trace of $A_D(G)$, then $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = |D|$.

(ii) Similarly the sum of squares eigenvalues of $A_D(G)$ is the trace of $(A_D(G))^2$. Then

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + \sum_{i \neq j}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j}^{n} a_{ij}^2 \\ &= |D| + 2m. \end{aligned}$$

□

Bapat and S.Pati [3] proved that if the graph energy is a rational number then it is an even integer.Similar result for minimum dominating energy is given in the following theorem.

Theorem 2.4. *Let G be a graph with a MMD set. If the MMD energy $E_D(G)$ of G is a rational number, then*

$$E_D(G) \equiv |D| \pmod 2.$$

Proof. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be MMD eigenvalues of a graph G of which $\lambda_1, \lambda_2, ..., \lambda_r$ are positive and the rest are non-positive, then

$$\begin{aligned} \sum_{i=1}^{n} |\lambda_i| &= (\lambda_1 + \lambda_2 + ... + \lambda_r) - (\lambda_{r+1} + \lambda_{r+2} + ... + \lambda_n). \\ &= 2(\lambda_1 + \lambda_2 + ... + \lambda_r) - (\lambda_1 + \lambda_2 + ... + \lambda_n). \\ &= 2q - |D|. \; Where \; q = \lambda_1 + \lambda_2 + ... + \lambda_r. \end{aligned}$$

Therefore, $E_D(G) = 2q - |D|$, and the proof is completed. □

3. Minimum Maximal Domination Energy of Some Standard Graphs

In this section, we investigate the exact values of the MMD energy of some standard graphs.

Theorem 3.1. *For the complete graph K_n, $n \geq 2$,* $\quad E_D(K_n) = n$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. Then $\gamma_m = n$. Hence, the MMD set of a complete graph is $D = \{v_1, v_2, \cdots, v_n\}$. Therefore, the MMD matrex is

$$A_D(K_n) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{n\times n}$$

The respective characteristic polynomial is

$$f_n(K_n, \lambda) = \begin{vmatrix} \lambda-1 & -1 & \cdots & -1 \\ -1 & \lambda-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda-1 \end{vmatrix}_{n\times n}$$

$$= \lambda^{n-1}(\lambda - n).$$

The MMD spectrum of K_n will be written as

$$MMD\ Spec(K_n) = \begin{pmatrix} 0 & n \\ n-1 & 1 \end{pmatrix}$$

Therefore, the MMD energy of a complete graph is $\quad E_D(K_n) = n$.

□

Theorem 3.2. *For the complete bipartite graph $K_{r,s}$, $r \leq s$, the MMD energy is at most $(r+1) + 2\sqrt{rs-1}$.*

Proof. For the complete bipartite graph $K_{r,s}$, $(r \leq s)$ with vertex set $V = \{v_1, v_2, \cdots, v_r, u_1, u_2, \cdots, u_s\}$. The MMD set is $D = \{v_1, v_2, \cdots, v_r, u_1\}$. Then

$$A_D(K_{r,s}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(r+s)\times(r+s)}$$

The characteristic polynomial of $A_D(K_{r,s})$, where $n = r + s$ is

$$f_n(K_{r,s},\lambda) = \begin{vmatrix} \lambda-1 & 0 & 0 & \cdots & 0 & -1 & -1 & -1 & \cdots & -1 \\ 0 & \lambda-1 & 0 & \cdots & 0 & -1 & -1 & -1 & \cdots & -1 \\ 0 & 0 & \lambda-1 & \cdots & 0 & -1 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda-1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & -1 & \cdots & -1 & \lambda-1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & \lambda & 0 & \cdots & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= \lambda^{s-1}(\lambda-1)^{r-1}\left[\lambda^3 - 2\lambda^2 - (rs-1)\lambda + r(s-1)\right].$$

By analysing the last factor of $f_n(K_{r,s},\lambda)$ we get

$$\begin{aligned} f_n(K_{r,s},\lambda) &= \lambda^{s-1}(\lambda-1)^{r-1}\left[\lambda^3 - 2\lambda^2 - (rs-1)\lambda + r(s-1)\right] \\ &= \lambda^{s-1}(\lambda-1)^{r-1}\left[\lambda^2(\lambda-2) - (rs-1)(\lambda-2) - (rs+r-2)\right] \\ &\le \lambda^{s-1}(\lambda-1)^{r-1}\left[(\lambda-2)(\lambda^2-(rs-1))\right]. \end{aligned}$$

Hence,

$$f_n(K_{r,s},\lambda) \le \lambda^{s-1}(\lambda-1)^{r-1}(\lambda-2)(\lambda-\sqrt{rs-2})(\lambda-\sqrt{rs-1}).$$

it follows that

$$MMD\ Spec(K_{r,s}) \approx \begin{pmatrix} 0 & 1 & 2 & -\sqrt{rs-1} & \sqrt{rs-1} \\ s-1 & r-1 & 1 & 1 & 1 \end{pmatrix}$$

Therefore, the MMD energy of a complete bipartite graph is

$$E_D(K_{r,s}) \le (r+1) + 2\sqrt{rs-1}.$$

The equality holds if $r = s = 1$. □

Theorem 3.3. *For $n \ge 2$, the MMD energy of a star graph $K_{1,n-1}$ is at most $2 + 2\sqrt{n-2}$. The equality holds if and only if $n = 2$.*

Proof. Let $K_{1,n-1}$ be a star graph with vertex set $V = \{v_0, v_1, v_2, \cdots, v_{n-1}\}$, v_0 is the center, and the MMD set is $D = \{v_0, v_1\}$. Then

$$A_D(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}$$

The characteristic polynomial of $A_D(K_{1,n-1})$ is

$$f_n(K_{1,n-1},\lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda-1 & 0 & \cdots & 0 \\ -1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= \lambda^{n-3}\left[\lambda^3 - 2\lambda^2 - (n-2)\lambda + (n-2)\right].$$

By analysing the last factor of $f_n(K_{1,n-1},\lambda)$ we get

$$\begin{aligned} f_n(K_{1,n-1},\lambda) &= \lambda^{n-3}\left[\lambda^3 - 2\lambda^2 - (n-2)\lambda + (n-2)\right] \\ &= \lambda^{n-3}\left[\lambda^3 - 2\lambda^2 - (n-2)\lambda + 2(n-2) - (n-2)\right] \\ &\leq \lambda^{n-3}\left[\lambda^3 - 2\lambda^2 - (n-2)\lambda + 2(n-2)\right] \\ &= \lambda^{n-3}\left[(\lambda-2)(\lambda^2 - (n-2))\right] \\ &= \lambda^{n-3}(\lambda-2)(\lambda-\sqrt{n-2})(\lambda+\sqrt{n-2}) \end{aligned}$$

It follows that the MMD spectrum is

$$MMD\ Spec(K_{1,n-1}) \approx \begin{pmatrix} 0 & 2 & -\sqrt{n-2} & \sqrt{n-2} \\ n-3 & 1 & 1 & 1 \end{pmatrix}$$

Therefore, the MMD energy of a star graph

$$E_D(K_{1,n-1}) \leq 2 + 2\sqrt{n-2}.$$

□

The cocktail party graph, denoted by $K_{2\times p}$, is a graph having vertex set $V(K_{2\times p}) = \bigcup\limits_{i=1}^{p}\{u_i, v_i\}$ and edge set $E(K_{2\times p}) = \{u_iu_j, v_iv_j, u_iv_j, v_iu_j :\ 1 \leq i < j \leq p\}$. i.e. $n = 2p$, $m = \frac{p^2-3p}{2}$ and for ever $v \in V(K_{2\times p})$, $d(v) = 2p-2$.

Theorem 3.4. *For the cocktail party graph of order $n = 2p$, $p \geq 3$, the MMD energy is less than $(4p-5) + 2\sqrt{2p-1}$.*

Proof. Let $K_{2\times p}$ be the cocktail party graph having vertex set $V(K_{2\times p}) = \bigcup\limits_{i=1}^{p}\{u_i, v_i\}$. Then the maximal dominaion number of $K_{2\times p}$ is $\lambda_m(K_{2\times p}) = n-1$. And the MMD set of cocktail party graph is $D = \bigcup\limits_{i=1}^{p}\{u_i, v_i\} - \{v_p\}$. Hence, the MMD matrix of cocktail party graph is

$$A_D(K_{2\times p}) \begin{pmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 \end{pmatrix}_{2p\times 2p}$$

The characteristic polynomial of $A_D(K_{2\times p})$ is

$$f_p(K_{2\times p},\lambda)=\begin{vmatrix} \lambda-1 & 0 & -1 & -1 & \cdots & -1 & -1 \\ 0 & \lambda-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \lambda-1 & 0 & \cdots & -1 & -1 \\ -1 & -1 & 0 & \lambda-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \cdots & \lambda-1 & 0 \\ -1 & -1 & -1 & -1 & \cdots & 0 & \lambda \end{vmatrix}_{2p\times 2p}$$

$$=(\lambda-1)^{p-1}(\lambda+1)^{p-2}\left[\lambda^3-(2p-2)\lambda^2-(2p-1)\lambda+(2p-2)\right]$$

By analysing the last factor of $f_n(K_{1,n-1},\lambda)$ we get

$$\begin{aligned} f_n(K_{1,n-1}) &= (\lambda-1)^{p-1}(\lambda+1)^{p-2}\left[\{(\lambda-(2p-2))(\lambda^2-(2p-1))-(2p-2)^2\}\right] \\ &< (\lambda-1)^{p-1}(\lambda+1)^{p-2}\left[(\lambda-(2p-2))(\lambda^2-(2p-1))\right]. \end{aligned}$$

Therefore,

$$MMD\ Spec(K_{2\times p})\approx\begin{pmatrix} -1 & 1 & 2p-2 & -\sqrt{2p-1} & \sqrt{2p-1} \\ p-2 & p-1 & 1 & 1 & 1 \end{pmatrix}$$

Hence, the MMD energy of cocktail party graph is

$$E_D(K_{2\times p})<(4p-5)+2\sqrt{2p-1}.$$

□

4. Bounds for Minimum Maximal Domination Energy of a Graph

In this section we shall investigate with some bounds for MMD energy of graphs.

Theorem 4.1. *Let G be a connected graph of order n and size m. Then*

$$\sqrt{2m+\gamma_m}\le E_M(G)\le\sqrt{n(2m+\gamma_m)}$$

Proof. Consider the Couchy-Schwartiz inequality

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2\le\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right).$$

By choose $a_i=1$ and $b_i=|\lambda_i|$, we get

$$\begin{aligned} (E_D(G))^2=\left(\sum_{i=1}^{n}|\lambda_i|\right)^2 &\le\left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}\lambda_i^2\right) \\ &\le n(2m+|D|) \\ &\le n(2m+\gamma_m). \end{aligned}$$

Therefore, the upper bound is hold. For the lower bound, since

$$\left(\sum_{i=1}^{n}|\lambda_i|\right)^2 \geq \sum_{i=1}^{n}\lambda_i^2.$$

Then

$$(E_D(G))^2 \geq \sum_{i=1}^{n}\lambda_i^2 = 2m + |D| = 2m + \gamma_m.$$

Therefore.

$$E_D(G) \geq \sqrt{2m + \lambda_m}.$$

□

Similar to McClellands [14] bounds for energy of a graph, bounds for $E_D(G)$ are given in the following theorem.

Theorem 4.2. *Let G be a connected graph of order and size n, m respectively. If $P = det(A_D(G))$, then*

$$E_D(G) \geq \sqrt{2m + \gamma_m + n(n-1)P^{2/n}}.$$

Proof. Since

$$(E_D(G))^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)\left(\sum_{i=1}^{n}|\lambda_i|\right) = \sum_{i=1}^{n}|\lambda_i|^2 + 2\sum_{i\neq j}|\lambda_i||\lambda_j|.$$

Employing the inequality between the arithmetic and geometric means, we get

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{1/[n(n-1)]}.$$

Thus

$$\begin{aligned}(E_D(G))^2 &\geq \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{1/[n(n-1)]}\\ &\geq \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\left(\prod_{i=j}|\lambda_i|^{2(n-1)}\right)^{1/[n(n-1)]}\\ &= \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\left|\prod_{i\neq j}\lambda_i\right|^{2/n}\\ &= 2m + \gamma_m + n(n-1)P^{2/n}.\end{aligned}$$

Then the proof is completed. □

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