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The Open Neighborhood Number of a Graph

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Abstract—*Let G be a graph. A subset S of vertices in a graph G is an open neighborhood set if* $\boldsymbol{G} = \cup_{\boldsymbol{v \in S}} \langle \boldsymbol{N(v)} \rangle$ *where* $\boldsymbol{N(v)}$ *denotes an open neighborhood of a vertex* $\boldsymbol{v}$*. The minimum cardinality of an open neighborhood set is called the open neighborhood number of a graph, denoted by* $\boldsymbol{n_{op}(G)}$*. In this paper, we initiate the study of the open neighborhood number. We determine this number for some standard family of graphs and some bounds are obtained. Further we study the effect of the operation maximum degree based vertex addition on this parameter.*

Keywords— *Open Neighborhood set, Open Neighborhood number, Edge lifting.*

I. Introduction

Let $G = (V, E)$ be any graph. The concept of neighborhood number of a graph was introduced by E. Sampathkumar and Prabha S. Neeralagi [3] in 1985. In the article[3], they obtained some bounds and relationship of the parameter with other known graph theoretic parameters. In 1988, P.P. Kale [1]correct some of the results mentioned in the article [3]. Further study on this parameter was done by V.R. Kulli [2] in 1992. Motivated by this, in this article, we are initiating the study of the graph parameter open neighborhood number of a graph. As usual, throughout this article, we assume that by a graph we mean a finite, undirected graph without loops and multiple edges.

For each vertex $v \in V$, the open neighborhood of v is the set $N(v)$ containing all the vertices u adjacent to v and the closed neighborhood of v is the set $N[v] = N(v) \cup \{ v \}$. Let S be any subset of V, then the open neighborhood of S is $N(S) = \cup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$.

The minimum and maximum of the degree among the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$ respectively. A graph G is said to be regular if $\delta(G) = \Delta(G)$. A vertex v of a graph G is called a cut vertex if its removal increases the number of components. A bridge or cut edge of a graph is an edge whose removal increases the number of components. A vertex of degree one is called a pendant vertex. An edge incident to a pendant vertex is called a pendant edge. The graph containing no cycle is called a tree. A complete bi-partite graph $K_{1,3}$ is a tree called as *claw*. Any graph containing no sub-graph isomorphic to $K_{1,3}$ is called a claw-free graph.

II. The Open Neighborhood Number of a Graph

In this section, we define the neighborhood number of a graph and calculate the number for some standard family of graphs.

Definition2.1*: A subset S of vertices in a graph G is an open neighborhood set if* $G = \langle \cup_{v \in S} N(v) \rangle$ *where* $N(v)$ *denotes an open neighborhood of a vertex* v*. The minimum cardinality of an open neighborhood set is called the open neighborhood number of a graph, denoted by* $n_{op}(G)$*. Any open neighborhood set of cardinality* $n_{op}(G)$ *is called an* n_{op}*-set.*

From the definition of the open neighborhood number, it follows that the open neighborhood number is defined only for connected graphs of order at least two. Therefore, we assume that by a graph in G, we mean a connected graph of order at least two.

Observations:

1. Let G be a complete graph. Then $n_{op}(G) = 2$.
2. Let G be a star or a wheel graph. Then $n_{op}(G) = 2$.
3. Suppose $x = uv$ is an edge in G such that $N(u) \cap N(v) \geq n, then\ n_{op}(G) = 2$.

Theorem 2.2*: Let G be a Cycle or a Path with* $n \geq 3$ *vertices. Then* $n_{op}(G) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & if\ n \equiv 1\ (mod\ 4); \\ 2\left\lceil \frac{n}{4} \right\rceil & otherwise. \end{cases}$

Proof: Let G be a path or a cycle of order $n \geq 3$. Fisrt, suppose $n \equiv 1\ (mod\ 4)$. Since no vertex in G is adjacent to itself, for each vertex there must be one more vertex from its neighborhood to cover it. Therefore, $n_{op}(G) \geq \left\lceil \frac{n}{2} \right\rceil$. On the other hand, $\{ v_2, v_3, v_6, v_7, \dots, v_{4k-2}, v_{4k-1}, v_{4k}\}$ will be a open neighborhood set of $G$of cardinality $n_{op}(G) = \left\lceil \frac{n}{2} \right\rceil$.

Next, assume that $n \not\equiv 1 (mod\ 4)$. Clearly $\{v_2, v_3, \dots, v_{4k-1}, v_{4k}\}$ is an open neighborhood set of cardinality $2\left\lceil \frac{n}{4} \right\rceil$. Let D be any open neighborhood set of cardinality less than $2\left\lceil \frac{n}{4} \right\rceil$. As in the above case, since no vertex is adjacent to itself, the graph induced by that vertex does not contain that vertex itself. Further, $\left\lceil \frac{n}{2} \right\rceil$ vertices induces the graph not containing themselve. Thus, D cannot be a open neigborhood set. This proves that $n_{op}(G) = 2\left\lceil \frac{n}{2} \right\rceil$.

Definition2.3 [4]**:** For any two positive integers m, n with $m \geq 2$, the Jahangir graph $J_{n,m}$ is a graph of order $nm + 1$, consisting of a cycle of order nm with one vertex adjacent to exactly m vertices of C_{nm} at a distance n to each other. Jahangir graph $J_{2,16}$ is shown in figure 1.

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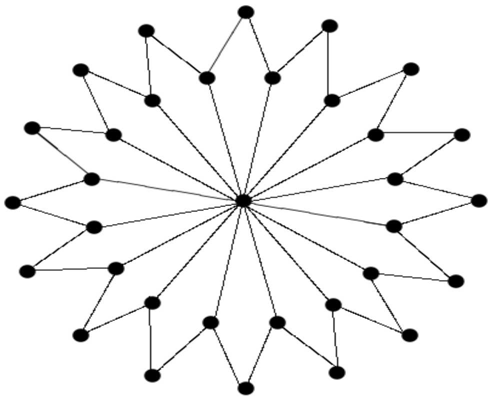

Fig 1: $J_{2,16}$

Theorem2.4: Let $G \cong J_{2,m}$ be a Jahangir graph. Then $n_{op}(G) = \left\lceil\frac{m}{2}\right\rceil + 1$.

Proof: Let $G \cong J_{2,m}$ be a Jahangir graph of order $2m+1$ and let $V(G) = \{v_1, v_2, \dots, v_{2m}, v_{2m+1}\}$ such that $v_{2m+1}$is the vertex at the center adjacent to exactly $m$vertices. Clearly, the set $\{v_1, v_5, \dots, v_{2m-3}, v_{2m}\}$ is an open neighborhood set of cardinality $\left\lceil\frac{n}{2}\right\rceil + 1$. Therefore $n_{op}(G) \leq \left\lceil\frac{m}{2}\right\rceil + 1$. On the other hand, assume that $D$is an open neighborhood set of G. Suppose that $v \notin D$. Then, the open neighborhood number of G co-incides with that of cycle of order $2m$. Thus, $n_{op}(G) \geq \left\lceil\frac{2m}{3}\right\rceil$, which is not possible. Therefore, we must have $v \in D$. Further, as $N(v_{2m+1})$ covers m vertices, its removal splits G into m components each isomorphic to K_1. Selecting one vertex from every two components, it follows that $G \cong N(D)$. Therefore, $n_{op}(G) = \left\lceil\frac{m}{2}\right\rceil + 1$.

Theorem2.5: Let $G \cong J_{3k+1,m}$ be a Jahangir graph. Then $n_{op}(G) = \frac{m(n-1)}{3} + 1$.

Proof: Let $G \cong J_{n,m}$ be a Jahangir graph with $m, n \geq 3$ and let $V(G) = \{v_1, v_2, \dots, v_{nm}, v_{nm+1}\}$, where v_{nm+1} is the vertex at the center, adjacent to vertices of C_{nm}. Assume that $n \equiv 1 \ (mod\ 3)$ i.e., $n = 3k+1$, for some positive integer k. From the definition of Jahangir graphs, the vertex v_{nm+1} is adjacent to m vertices of C_{nm} at a distance $3k+1$. Removing the vertex v_{nm+1} from G, the graph induced by $V(G) - \{v_{nm+1}\}$ splits into m components each component isomorphic to P_{3k}. Therefore, the minimum open neighborhood set of G is obtained by taking open neighborhood set from each component together with v_{nm+1}. That is, if $S = \cup_{i=1}^{m} S_i$, where S_i denotes n_{op}-set of i^{th} component, then S $\cup \{v_{nm+1}\}$ will be a minimum open neighborhood set of G. Since any vertex not in $S \cup \{v_{nm+1}\}$ will be adjacent to exactly one vertex in $S \cup \{v_{nm+1}\}$, no proper subset will be dominating set in G. Thus, $n_{op}(G) = \frac{m(n-1)}{3} + 1$.

Theorem2.6: *Let $G \cong K_{m_1,m_2,\dots,m_r}$ be a complete r −partite graph with $m_1 \leq m_2 \leq \dots \leq m_r$. Then $n_{op}(G) = 2$.*

Proof: Let $G \cong K_{m_1,m_2,\dots,m_r}$ be a complete multipartite graph with $m_1 \leq m_2 \leq \dots \leq m_r$. Let u, v be any two vertices taken from V_i and V_j for $i \neq j$. Then clearly, the graph induced by $N(u,v)$ will be G and so $n_{op}(G) \leq 2$. Since$n_{op}(G) \geq 2$ always, it follows that $n_{op}(G) = 2$.

Corollary 2.7: Let G be a complete bipartite graph, then $n_{op}(G) = 2$.

Definition 2.8: A firefly graph $F_{s,t,n-2s-2t-1}$ $(s \geq 0, t \geq 0, n-2s-2t-1 \geq 0)$ is a graph on n vertices having s triangles, t pendant paths of length 2 and $n-2s-2t-1$ pendant edges sharing a common vertex.

Let F_n be the set of all Firefly graphs $F_{s,t,n-2s-2t-1.}$ Note that F_n contains the stars S_n $(\cong F_{0,0,n-1})$, stretched stars $(\cong F_{0,t,n-2t-1})$, friendship graphs$(\cong F_{\frac{n-1}{2}}, 0, 0)$ and butterfly graphs $(\cong F_{s,0,n-2s-1})$.

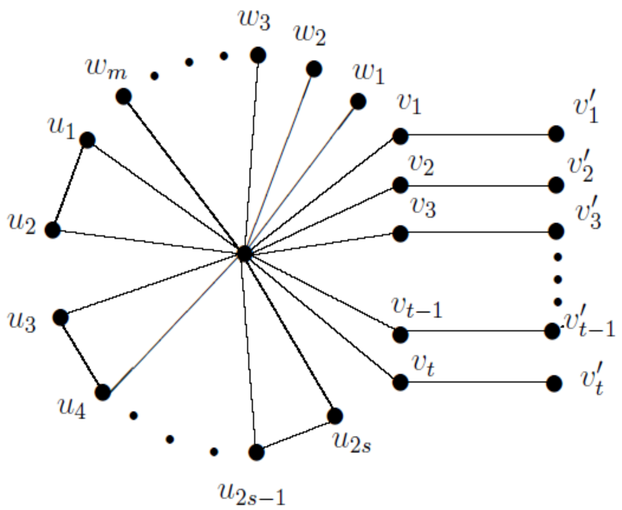

Figure 2: Firefly graph

Theorem2.9: Let $G \cong F_{s,t,n-2s-2t-1}$be a firefly graph. Then $n_{op}(G) = t+1$.

Proof: Let $G \cong F_{s,t,n-2s-2t-1}$be a firefly graph with t pendant paths. Let v_n be a vertexcommon to triangles, pendant paths of length 2 and the pendant edges. Then, the neighbourhood of v_n includes $V(G)$ except the leaves of pendant paths. Thus, v_n together with all leaves of pendant paths will be an open neighborhood set of G. Therefore, $n_{op}(G) = t+1$.

Proposition2.10: *Let G_1 and G_2 be any two graphs. Then $n_{op}(G_1 \vee G_2) = 2$.*

Proof: Let G_1 and G_2 be any two graphs. From the definition of join of two graphs, it follows that each vertex in $V(G_1)$ is adjacent to every vertex in $V(G_2)$. Hence, taking vertex u from $V(G_1)$ and a vertex v from $V(G_2)$, the set $\{u, v\}$ will be a open neighborhood set of $G_1 \vee G_2$ and so $n_{op}(G_1 \vee G_2) = 2$.

Proposition2.11: *Let G be any graph of order n and having no isolated vertices. Then $n_{op}(G \circ H) = n$, for any graph H.*

Proof: Let G be a graph having no isolated vertex and H be

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any graph. As the copy of a graph H is attached to the vertices of G, it follows that V(G) itself a minimal open neighborhood set of $G \circ H$ and so $n_{op}(G) \leq n$.On the other hand, for any vertex $v \in V(G)$, the set $V(G) - \{v\}$ will not be an open neighborhood set as the graph induced by $V(G) - \{v\}$ is not same as $G \circ H$. Hence $n_{op}(G) \geq n$ and so $n_{op}(G) = n$.

III. Some Bounds for $n_{op}(G)$

Theorem3.1: *Let G be any graph. Then, $2 \leq n_{op}(G) \leq n$. Further, $n_{op}(G) = 2$ if and only if G contains an edge of degree at least $n-2$.*

Proof: Let G be a graph of order n. For any vertex $v \in V(G)$, the open neighborhood of v does not contain the vertex v itself. Hence any open neighborhood set must be of order at least two, proving the lower bound and upper bound holds trivially.

Proposition3.2: Let G be a connected graph with $n \geq 2$ vertices. Then $n_{op}(G) = n$ if and only if $G \cong P_2$.

Proof: Let G be a connected graph and assume that $n_{op}(G) = n$. Suppose $n \geq 3$, since G connected there exists a vertex v such that $V(G) - \{v\}$ is an open neighborhood set in G and hence $n_{op}(G) \leq n-1$, a contradiction. This contradiction shows that $n = 2$. As the graph G is connected, we must have $G \cong P_2$

Corollary3.3: Let G be any graph. Then $n_{op}(G) = n$ if and only if each component of G is isomorphic to P_2.

Theorem3.4: *Let G be any graph. Then $\gamma(G) \leq n_0(G) \leq n_{op}(G) \leq \gamma_{pd}(G) \leq 2\gamma(G)$.+*

Proof: Let G be any graph. For an arbitrary vertex $v \in V(G)$, we have $N(v) \subset N[v]$and therefore, we should have $n_0(G) \leq n_{op}(G)$. Since in an open neighborhood set, each vertex $v$contains a neighbor vertex, it follows that every vertex in a n_{op} −set possesses a backup. Therefore, we have $n_{op}(G) \leq \gamma_{pd}(G) \leq 2\gamma(G)$.

IV. Effect of Maximum Degree Based Vertex Addition

In this section, we study the effect of maximum degree based vertex addition on the open neighborhood number of a graph and we partition the class of graphs depending on the effect.

When we study the graph theoretic parameters it also important to study the behavior of the parameter when the graph under consideration is modified by applying the graph operations such as vertex or edge addition, removal of a vertex or an edge, edge splitting etc. In this section, we study the effect of the operation called as maximum degree based vertex addition on the open neighborhood number of a graph.

Definition4.1: Suppose u and v be any two non-adjacent vertices in G, then the process of inserting a vertex w and edges uw and vw whenever $deg\, u + deg\, v \leq \Delta(G)$ is called the maximum degree based vertex addition.

The graph obtained on applying this operation once for the given graph will be denoted by G_w^*. It clear from the definition that $V(G_w^*) = V(G) \cup \{w\}$ and $E(G_w^*) = E(G) \cup \{uw, vw\}$. Therefore, this operation increases the order of the graph by one and size by two.

To illustrate the effect of maximum degree based vertex addition on the open neighborhood number of a graph, consider a bi-star $G \cong B(m,n)$ with $m, n \geq 2$. Then, $n_{op}(G) = 2$, whereas $n_{op}(G_w^*) = 3$, for any pair of non-adjacent vertices in G. Let G be a path P_3 with 3 vertices. Then $n_{op}(G) = 2$ and $G_w^* \cong C_4$ and hence $n_{op}(G_w^*) = 2$.

Thus, the new operation may increase the open neighborhood number of a graph or it may leave unaltered. From the above illustrations, it is proved that the operation maximum degree based vertex addition may increase the value of $n_{op}(G)$ or remain unchanged. But this operation does not decrease the value of $n_{op}(G)$. Thus, it is possible to partition the class $\mathcal{F}$ of all graphs into two sets $\mathcal{F}^+$ and $\mathcal{F}^0$ where

$$\mathcal{F}^+ = \{ G \in \mathcal{F} \,|n_{op}(G_w^*) > n_{op}(G);$$
$$\mathcal{F}^0 = \{ G \in \mathcal{F} |n_{op}(G_w^*) = n_{op}(G).$$

Further, from the above examples it follows that the sets $\mathcal{F}^0$ and $\mathcal{F}^+$ are non-empty.

Partitioning the vertex set of a graph G into subsets of its vertex set having certain property is also one of the directions for the research in graph theory. For example, one such partition is domatic partition which is a partition of $V(G)$ into dominating sets. Analogously, we can demand each subset in the partition of $V(G)$ to have the property being an open neighborhood set/neighborhood set instead of the parameter domination alone. We may call this partition of $V(G)$ an open neighborhood number partition, similarly, neighborhood number partition.

Neighborhood set partition exists for all graphs whereas open neighborhood partition exists only for connected graphs of order at least 2. The maximum cardinality of a open neighborhood partition set is called open neighborhood partition number denoted by $N_{opn}(G)$. Now, begin investigating the parameter $N_{opn}(G)$.

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