INTERNATIONAL JOURNAL OF PURE AND APPLIED RESEARCH IN ENGINEERING AND TECHNOLOGY

A PATH FOR HORIZING YOUR INNOVATIVE WORK

THE INDEPENDENT TRANSVERSAL PAIRED DOMINATION IN GRAPHS

K. B. MURTHY[1], NAYAKA S. R[2]

1. Department of Statistics and Applied Mathematics, College of Agriculture, UAS, GKVK Campus, Bengaluru-560065, India
2. Department of Mathematics, P.E.S College of Engineering, Mandya-571401, India.

Accepted Date: 03/12/2015; Published Date: 01/01/2016

Abstract: - Let $G = (V, E)$ be a graph, a set $S \subseteq V$ is called a dominating set in G if every vertex in $V - S$ is adjacent to a vertex in S. A dominating set S is called a paired dominating set in G if the induces subgraph $\langle S \rangle$ contains at least one perfect matching. A paired dominating set which intersecting every maximum independent set in G is called an independent transversal paired dominating set in G. The minimum cardinality of an independent transversal paired dominating set is called the independent transversal paired domination number of G, denoted by $\gamma_{itp}(G)$. In this paper, we begin to study this parameter. Exact values of $\gamma_{itp}(G)$ for some families such as flower graph, prism graphs and product graphs are obtained. Further some bounds are estimated for $\gamma_{itp}(G)$ and also we study the effect of the graph operation called maximum degree based vertex addition. **2010 Mathematics Subject Classification: Primary 05C50, 05C69**.

Keywords: Paired dominating set, Independent set, Independent transversal paired dominating set, Independent transversal paired domination number.

Corresponding Author: MR. K. B. MURTHY

Access Online On:

www.ijpret.com

How to Cite This Article:

K. B. Murthy, IJPRET, 2015; Volume 4 (5): 13-27

PAPER-QR CODE

INTRODUCTION

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. For any graph theoretic terminology, we refer to the book by Chartrand and Lesniak [2]. All graphs in this paper are assumed to be non- trivial. One of the fastest growing areas in graph theory is the study of domination and related subset problems such as independence, covering and matching. In fact, there are scores of graph theoretic concepts involving domination, covering and independence. The bibliography in domination maintained by Haynes et al. [6] currently has over 1200 entries; Hedetniemi and Laskar [9] edited a recent issue of Discrete Mathematics devoted entirely to domination, and a survey of advanced topics in domination is given in the book by Haynes et al. [7].

Nevertheless, despite the many variations possible, we can so far identify only a limited number of basic domination parameters; basic in the sense that they are defined for every non-trivial connected graph. For instance, independent domination, connected domination, total domination, global domination and acyclic domination are some basic domination parameters. In this sequence, we introduce another basic domination parameter namely independent transversal paired domination motivated by independent transversal domination introduced by Ismail Sahul Hamid [10] and initiate the study of this new domination parameter.

Let G be any graph with n vertices and m edges. The open neighbourhood of a vertex $v \in V(G)$ is denoted and defined by $N(v) = \{u \in V / uv \in E\}$, the set of vertices adjacent to v. The closed neighbourhood is denoted and defined by $N[v] = N(v) \cup \{v\}$. For any subset S of G, the open and closed neighbourhoods of S in G is defined by $N(v) = \cup_{v\in S} N(v)$ and $N(v) = \cup_{v\in S} N[v]$ The subgraph induced by a set $S \subseteq V$ is denoted $\langle S\rangle$. If G is a graph, then G+ is the graph obtained from G by attaching a pendant edge at every vertex of G.

A set $S \subseteq V$ is called a dominating set of G if every vertex in $V \setminus S$ is adjacent to at least one vertex of S i.e., S is a dominating set of G if $N[S] = V(G)$ and the minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. A minimum dominating set of a graph G is called a γ-set of G. An independent dominating set of G is a dominating set S of G such that S is an independent set in G. The independent domination number $i(G)$ is the cardinality of the minimum independent dominating set. The maximum cardinality of an independent set is called the independence number of G and is denoted by $\beta_0(G)$. A maximum independent set is called a β_0-set.

A subset M of E is called a matching of G if no two edges in M are incident in G. The two ends of an edge are said to be matched under M. If every vertex of G is matched under M, then M is called a perfect matching. M is said to be a maximum matching of G if no subset of G containing M properly is a matching of G. Clearly, every perfect matching is maximum. A dominating set S such that $\langle S \rangle$ contains a perfect matching is called a paired dominating set. A total dominating set is a dominating set S such that $\langle S \rangle$ has no isolated vertices. The minimum cardinality of a paired (total) dominating set is called the paired (total) domination number and is denoted by $\gamma_{pd}(G)$ ($\gamma_t(G)$). These two parameters can respectively be seen in [3], [11] and [12].

Definition 1.1. *A paired dominating set $S \subseteq V$ of a graph G is called as an independent transversal paired dominating set if S intersects every maximum independent set of G.*

The minimum cardinality of an independent transversal paired dominating set of G is called the independent transversal domination number of G and is denoted by $\gamma_{itp}(G)$. An independent transversal paired dominating set S of G with $|S| = \gamma_{itp}(G)$ is called a γ_{itp}-set.

Definition 1.2.[14] *The join of two graphs G_1 and G_2, denoted by $G = G_1 \vee G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1) \text{ and } v \in V(G_2)\}$.*

Definition 1.3.[14] *The corona* of two G_1 *and* G_2 *is defined as the graph* $G = G_1 \circ G_2$ *formed by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, where i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.*

Definition 1.4.[1] T*he Cartesian product of simple graphs G and H is the graph $G \square H$, whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G)$ and $v_1 = v_2$ and $v_1v_2 \in E(H)$ or $u_1 = u_2$ and $v_1v_2 \in E(G)$.*

Definition 1.5.[4] *A graph G is called an $(n \times m)$-flower graph if it has the following set of vertices $V(G) = \{1, 2, , n, n+1, n(m-1)\}$ and the edge set $E\{G\} = \{\{1, 2\}, \{2, 3\}, \ldots\ldots, \{n-1, n\}, \{n, 1\}\} \cup \{\{1, n+1\}, \{n+1, n+2\}, \{n+2, n+3\}, \{n+3, n+4\}, \{n+m-3, n+m-2\}, \{n+m-2, 2\}\} \cup \{\{2, n+m-1\}, \{n+m-1, n+m\}, \{n+m, n+m+1\}, \{n+m+1, n+m+2\}, \ldots, \{n+2(m-2)-1, n+2(m-2)\}, \{n+2(m-2), 3\}\} \cup \ldots\ldots \cup \{\{n, n+(n-1)(m-2)+1\}, \{n+(n-l)(m-2)+1, n+(n-1)(m-2)+2\}, \{n+(n-1)(m-2)+2, \{n+(n-1)(m-2)+2, n+(n-1)(m-2)+3, \{n+(n-1)(m-2)+3\}, n+(n-1)m-2)+4\}, , \{nm-1, nm\}, \{nm, 1\}\}$.*

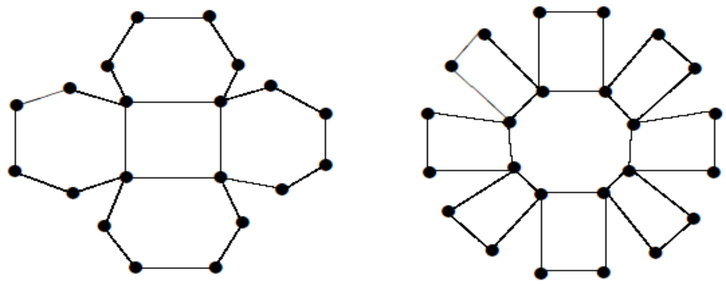

Figure 1.

In other words, a graph G is called a $(n \times m)$-flower graph if it has n vertices which form an n-cycle and n sets of $m - 2$ vertices which form m-cycles around the n cycle so that each m-cycle uniquely intersects with the n-cycle on a single edge. This graph will be denoted by $f_{n\times m}$. It is clear that $f_{n\times m}$ has $n(m - 1)$ vertices and nm edges. The m-cycles are called the petals and the n-cycle is called the center of $f_{n\times m}$. The n vertices which form the center are all of degree 4 and all the other vertices have degree 2.

Definition 1.6.[16] *The crown graph* S_n^0 for $n \geq 3$ *is the graph with vertex set*

V := {$u_1,u_2,u_3,...u_n,v_1,v_2,v_3,...v_n$} and an edge from {$u_iv_i$: $1 \leq i, j \leq n$; $i \neq j$}. Therefore S_n^0 *coincides with the complete bipartite graph $K_{m,n}$ with horizontal edges removed.*

Definition 1.7.[16] *The cocktail party graph* $K_{n\times 2}$, *is a graph having the vertex set* V: ={$u_l,u_2,u_3,...u_n,v_l,v_2,v_3,...,v_n$ } *and having the edge set* E : = {u_iu_j , v_iv_j , u_iv_j , v_iu_j / $1 \leq i < j \leq n$}.This graph is also called as complete *n*-partite graph.

Definition 1.8.[15] *The n-Sunlet graph is a graph of order* $2n$ *obtained by attaching n pendant edges to a cycle with* n *vertices. i.e., the graph* $C_n \circ K_1$ *is referred as a Sunlet graph.*

As usual given a real number x, $\lfloor x \rfloor$ denotes the greatest integer less than x and $\lceil x \rceil$ denotes the smallest integer greater than x.

2. Independent Transversal Paired Domination Number:

In this section, we determine the value of independent transversal domination number for some standard families of graphs such as paths, cycles and wheels. Also we determine $\gamma_{itp}(G)$ for union of connected component graphs and graphs obtained by applying graph operations such as join and product of graphs.

Observation 2.1. *If G is a complete multipartite graph having r maximum independent sets, then*

$$\gamma_{itp}(G) = \begin{cases} 2, & if\ r = 1; \\ 2\left\lceil \frac{r}{2} \right\rceil & otherwise. \end{cases}$$

Observation 2.2. *For any complete graph K_n with n even, $\gamma_{itp}(K_n) = n$.*

If n is odd, then the complete graph K_n contains no independent transversal paired dominating set. In view of the above observation we restrict our self to non-complete graphs with n vertices whenever n is odd at the study of independent transversal paired domination.

So in the rest of the paper we assume that by a graph we mean a graph G contains no isolated vertices and is non-complete if order of G is odd.

Proposition 2.3. *Let $G \cong K_{m,n}$ be a complete bipartite graph. Then $\gamma_{itp}(G)$ = 2.*

Proof. Let G be a complete bipartite graph with the vertex set $V(G)$= $\{v_1,v_2,\ldots\ldots,v_m,u_1,u_2,\ldots\ldots,u_n\}$. We may assume that $m \le n$. Then G contains one or two independent set according *as* $m < n$ *or* $m = n$. Clearly any edge $e = (u_i, v_j)$, for some i, j will be an independent transversal paired dominating set of G.Hence $\gamma_{itp}(G)$ =2.

Corlllary 2.4. *Let $G \cong K_{1,n-1}$ be a star with $n \ge 2$ vertices. Then $\gamma_{itp}(G)$ = 2.*

Proof. Let G be a star with n vertices and let $V(G)$={ $u_1,u_2,\ldots\ldots,u_n$} where u_n is a vertex at the center. Then clearly G contains a unique maximum independent set of order $n-1$ consisting of all the vertices of G except u_n. Also, any edge in G will is construct a minimum paired dominating set of G intersecting every maximum independent set of G. Thus any paired dominating set of G itself the independent transversal paired dominating of G.

Hence $\gamma_{itp}(G) = \gamma_{pd}(G)$ = 2.

Proposition 2.5. *For any bi-star B(m,n), $\gamma_{itp}(B(m,n))$ = 4, where $m, n \ge 2$*

Proof. Let $V(B(m,n))$ = $\{v_1,v_2,\ldots v_m,u_l,u_2,\ldots u_n,u,v\}$ where *u* and *v* are the vertices at the center. Then *B(m, n)* contains a unique maximum independent set containing all the pendent vertices. S = {*u, v*} is the minimum paired dominating set of $B(m,n)$, by adding one pendent vertex from each the stars $K_{1,n}$ and $K_{1,m}$ we obtain an independent transversal paired dominating set and so $\gamma_{itp}(B(m,n)) \le 4$. On the other hand, since $\gamma_{pd}(B(m,n))$ = 2 and γ_{pd} -set do not intersects the independent set of *B(m,n),* it follows that $\gamma_{itp}(B(m,n)) \geq 3$. Since γ_{itp} is an even integer always, we have $\gamma_{itp}(B(m,n)) \geq 4$. Therefore $\gamma_{itp}(B(m,n)) = 4$.

Proposition 2.6. *Let G be a crown graph with $2n$ vertices. Then $\gamma_{itp}(G) = 4$.*

Proof. Let G be a crown graph with the vertex set $V(G)=\{v_1, v_2, \ldots v_n, u_1, u_2, \ldots, u_n\}$.*Then* G contains two maximum independent sets of size *n.* Clearly the set $S = \{v_1, v_2, u_1, u_2\}$ is a paired dominating set of G intersecting both independent sets of G. Hence $\gamma_{itp} \leq 4$.

Also the minimum parird domination number for crown graph is 4 and S itself is minimum paired dominating set of G. Hence $\gamma_{itp} = 4$ and S is the minimum independent transversal paired dominating set of G.

Theorem 2.7. *For any path P_n of order n, we have*

$$\gamma_{itp}(P_n) = \begin{cases} 4, & if\ \ n = 4; \\ 2\left\lceil \frac{n}{4} \right\rceil & otherwise. \end{cases}$$

Proof. If $n = 4$, the obviously $\gamma_{itp}(P_4) = 4$. So let $n \neq 4$. For any path P_n with the vertices $\{v_1, v_2, \ldots, v_n\}$ any maximum independent set of P_n must be contains either v_1 or v_2 , we have two cases

<u>Case 1</u>: the maximum independent set contains v_1 so any minimum paired dominating set S in P_n either contains v_1, v_2 and it will intersect the maximum independent set in v_1 or v_2, v_3 and it will intersect the maximum independent set in v_3. Therefore any minimum paired dominating set in in P_n will intersect every maximum independent set in P_n. Hence, $\gamma_{itp}(P_n) = \gamma_{pd}(P_n) = 2\left\lceil \frac{n}{4} \right\rceil$.

<u>Case 2</u>: the maximum independent set contains v_2 in the same way as in case 1. It is easy to see that $\gamma_{itp}(P_n) = \gamma_{pd}(P_n) = 2\left\lceil \frac{n}{4} \right\rceil$.

Theorem 2.8. *For any cycle C_n with $n \geq 4$, we have*

$$\gamma_{itp}(C_n) = \begin{cases} 2, & if\ \ n = 4; \\ 2\left\lceil \frac{n}{4} \right\rceil & otherwise. \end{cases}$$

Proof: Let $V(C_n)=\{v_1, v_2, \ldots\ldots\ldots, v_n\}$.Clearly we have $\gamma_{itp}(C_4) = 2$.Assume $n \neq 4$. Then C_n contains two maximum independent sets and we note that every paired dominating set in C_n intersects the β_0-sets of C_n. Hence any paired dominating set in

C_n will be an independent transversal paired dominating set of C_n. Thus γ_{itp}-set of C_n will be same as γ_{pd}-set and so $\gamma_{itp}(C_n) = \gamma_{pd}(C_n) = 2\left\lceil\frac{n}{4}\right\rceil$.

Theorem 2.9. *For any wheel graph W_n on $n \geq 5$ vertices,*

$$\gamma_{itp}(W_n) = \begin{cases} 2, & if \ \ n = 5 ; \\ 4 & otherwise. \end{cases}$$

Proof : Let $V = \{v_1, v_2, \ldots\ldots, v_{n-1}, v_n\}$ be the vertex set of W_n and let v_n be the vertex at the center.

Case1: If $n = 5$, then $S = \{v_1, v_2\}$ is a paired dominating set of W_5 and as we saw in cycle we note that any independent set in W_5 must be contains one of $v_1$or v_2. Therefore, $S = \{v_1, v_2\}$ is an *independent* transversal paired dominating set of W_n. Hence $\gamma_{itp}(W_5) \leq 2$. Thus, $\gamma_{itp}(W_5) = 2$.

Case2: If $n \geq 6$, it is easy to observe that any maximum independent set in W_n contains either v_1 or v_2.Thus the set $D=\{v_1, v_2\}$ intersects every β_0-set of W_n. Adding the vertex v_n and a vertex v_j, with $j \notin \{1, 2\}$ to the set D, the resulting set will be an independent transversal paired dominating set of W_n and so $\gamma_{itp}(W_n) \leq 4$. Since W_n contains two disjoint β_0-sets, and $\gamma_{itp}(W_n)$ cannot be 2, because any minimum independent transversal paired dominating set of W_n must contains the center and one vertex from each maximum independent set, it follows that $\gamma_{itp}(W_n) \geq 3$. But γ_{itp} is an even integer always. Hence $\gamma_{itp}(W_n) = 4$.

Lemma 2.10. *Let $G \cong f_{n\times m}$ be flower graph with n vertices. Then*

$$\beta_0(G) = \begin{cases} \frac{n(m-1)}{2} - 1, & if \ \ n \ odd \ and \ \ m \ even \ ; \\ \frac{n(m-1)}{2} & otherwise. \end{cases}$$

Proof: Let G be a flower graph with $n(m-1)$ vertices. It is easy to note that degree of every vertex in G is an even integer. The maximum independent set of G is obtained by selecting alternative vertices of G. we have two cases:

Case 1: If n is odd and m is an even integer, then the order of G will be an odd integer and the set $S = \{v_1, v_3, \ldots\ldots, v_{n(m-1)-3}\}$will be the maximum independent set in G.

Therefore, $\beta_0(G) = \frac{n(m-1)}{2} - 1$.

Case 2: If n, m not as in case 1, the order of G will be an even integer. Further the set $S' = \{v_2, v_4, \ldots v_{n(m-1)}\}$will be the maximum independent set in G. Therefore, we have $\beta_0(G) = \frac{n(m-1)}{2}$.

Theorem 2.11. *Let $G \cong f_{n\times m}$ be flower graph n vertices. Then $\gamma_{itp}(G) = \frac{n(m-1)}{2}$.*

Proof: Let G be a flower graph with $n(m-1)$ vertices. Then G contains two maximum independent sets and every paired dominating set of G intersects both the sets. Hence the minimum independent transversal paired dominating set of G is same as the minimum paired dominating set of G. If n is odd and m is an even integer, then $S = \{v_1, v_2, v_5, v_6, \ldots, v_{n(m-1)-3}, v_{n(m-1)-2}, v_{n(m-1)-1}, v_{n(m-1)}\}$ will be the minimum paired dominating set of G. Further each edge in $\langle S\rangle$ as one of its end vertex from the maximum independent set of G and in this case S contains $\frac{\beta_0+1}{2}$ vertices from the maximum independent set of G. Hence $\gamma_{itp}(G) = \beta_0 + 1$. i.e., by using Lemma 2.9, $\gamma_{itp}(G) = \frac{n(m-1)}{2}$. For other values of n, m the set $S = \{v_1, v_2, v_5, v_6 \ldots \ldots, v_{n(m-1)-3}, v_{n(m-1)-2}\}$ will be the minimum paired dominating set of G. Since each edge in $\langle S\rangle$ contains an end vertex from β_0-set of G and contains exactly $\frac{\beta 0}{2}$ vertices. Since edges in $\langle S\rangle$ are non-adjacent, γ_{itp} will be twice the number of edges. But each edge in $\langle S\rangle$ corresponds to a vertex in β_0-set and totally there are $\frac{\beta_0}{2}$ such vertices.

Hence $\gamma_{itp}(G) = \beta_0$. i.e., by using the above Lemma 2.9, $\gamma_{itp}(G) = \frac{n(m-1)}{2}$.

Theorem 2.12. *If G is a union of m connected components $G_1, G_2, \ldots, G_m$, then $\gamma_{itp}(G) = min_{1\le i\le r}\ \gamma_{itp}(G_i) + \sum_{j=1, j\neq i}^{r} \gamma_{pd}(G_j)$.*

Proof. We first assume that $\gamma_{itp}(G_1) + \sum_{j=2}^{m} \gamma_{pd}(G_j) = min_{1\le i\le m}\{\gamma_{itp}(G) + \sum_{j=1,j\neq i}^{m} \gamma_{pd}(G_j)\}$. Let S be a γ_{itp}-set of G_1 and let S_j be a γ_{pd}-set of G_j, for all $j \ge 2$. Then $S \cup (\cup_{j=2}^{m} S_j)$ is an independent transversal paired dominating set of G and hence $\gamma_{itp}(G) \le \gamma_{itp}(G_1) + \sum_{j=2j}^{m} \gamma_{pd}(G_j) = min_{1\le i\le m}\{\gamma_{itp}(G_i) + \sum_{j=1,j\neq i}^{m} \gamma_{pd}(G_j)\}$.
Conversely, let S' be any independent transversal paired dominating set of G. Then S' must intersect the vertex set $V(G_j)$ of each component G_j of G and $S' \cap V(G_j)$ is a paired dominating set of G_j for all $j \ge 1$. Further, for at least one j, the set $S' \cap V(G_j)$ must be an independent transversal paired dominating set of G_j, for otherwise each component G_j will have a maximum independent set not intersecting the set $S' \cap V(G_j)$ and so the union of these maximum independent sets from a maximum independent set of G not intersecting S'.

Hence $|S'| \ge min_{1\le i\le m}\{\gamma_{itp}(G_j) + \sum_{j=1,j\neq i}^{m} \gamma_{pd}(G_j)\}$.

Theorem 2.13. *Let G be any cycle C_n on n vertices. Then $\gamma_{itp}(C_n \circ P_k) = 2\left\lfloor\frac{n}{2}\right\rfloor + 2$, where $k \ge 3$.*

Proof: Consider the graph C_n o P_k . By the definition of the corona of graphs, Let S_i be the maximum independent set of the path P_k joined to the vertex v_i of C_n. Then $S = \cup_{i=1}^{n} S_i$, consisting of n copies of S_i is a maximum independent set of G o P_k and $|S| = n\beta_0(P_k)$, where n is the number of vertices in C_n. let M be the maximum matching in C_n , we have two cases:

Case 1: If n is odd, there is a vertex v of C_n which is not matched under M. Choosing an edge e from the path P_k joined to the vertex v, we obtain the set $S' = M \cup \{e\}$ which is a paired dominating set of C_n o P_k, intersecting the β_0 - set of C_n o P_k. In fact S' is the minimum independent paired dominating set of C_n o P_k, since M is the minimum set in $V(C_n)$ covering all the vertices of C_n o P_k except the path P_k joined to the vertex v. Therefore, $\gamma_{itp}(C_n$ o $P_k) = |S|$ $= 2m + 2$, where $m = |M|$, matching number of G. Hence $\gamma_{itp}(C_n$ o $P_k) = 2\left\lfloor\frac{n}{2}\right\rfloor + 2$.

Case 2: If n is even, then the maximum matching of G itself is the minimum paired dominating set of C_n *o* P_k intersects no independent set of C_n o P_k. because $k \geq 3$. Hence choosing an edge e from the path P_k attached to any vertex of G, the set $S' = M \cup \{e\}$ is a γ_{itp}-set of C_n o P_k and so $\gamma_{itp}(C_n$ o $P_k) = 2\left\lfloor\frac{n}{2}\right\rfloor + 2$.

Theorem 2.14. *Let G be any connected graph with n vertices. Then* $\gamma_{itp}(G$ o $\overline{K_k}) = 2(m + 1)$ *where m is a matching number of G.*

Proof: Consider the graph G o $\overline{K_k}$. We have two possible cases here. The case n odd is similar to the proof of Theorem [2.12]. If n even then choose the maximum matching M of G. Delete an edge $e = uv$, where $e \in M$ and add the edges say e_1 and e_2 joining the vertices u and v of G respectively to $\overline{K_k}$. Then $S = M \setminus \{e\} \cup \{e_1, e_2\}$ is clearly the minimum independent transversal paired dominating set of G o $\overline{K_k}$.

Hence $\gamma_{itp}(G$ o $\overline{K_k}) = 2(m + 1)$.

Theorem 2.15. *Let G_1 and G_2 be any two graphs. If $\beta_0(G_i) \geq \beta_0(G_j)$, i, j = 1,2 . Then*

$$\gamma_{itp}(G_1 \vee G_2) = \gamma_{itp}(G_i).$$

Proof. Let G_1 and G_2 be any two graphs such that $\beta_0(G_1) \geq \beta_0(G_2)$. Since all the vertices of G_1 and G_2 are adjacent to each other, each vertex in G_1 dominates G_2. Clearly the independent sets of $G_1 \vee G_2$ are the independent sets of G_1 and G_2. Since $\beta_0(G_1) \geq \beta_0(G_2)$, the maximum independent set of $G_1 \vee G_2$ is the maximum independent set of G_1. Let S be a γ_{itp}-set of G_1 then clearly S also the γ_{itp}-set of $G_1 \vee G_2$. Hence, $\gamma_{itp}(G_1 \vee G_2) = \gamma_{itp}(G_i)$.

The grid graph is the graph Cartesian product $P_n \square P_m$ where P_n is the path graph with *n* vertices. For $m = 2$, the graph $P_n \square P_m$ is called the Ladder graph having *2n* vertices and *3n* - 2 edges.

Theorem 2.16. *Let P_n be a path with n vertices. Then* $\gamma_{itp}(P_n \square P_2) = 2\left\lceil \frac{n}{3} \right\rceil$.

Proof. Let $V(P_n \square P_2)$={ $v_1, v_2, v_3, \dots \dots \dots, v_n, u_1, u_2, u_3, \dots, u_n$}.Then the product graph $P_n \times P_2$ contains two maximum independent sets of size n.Any edge taken between two paths has one of its end vertex from the β_0-set. The set S={$v_2, u_2, v_5, u_5, \dots, v_{3k+2}, u_{3k+2}$} where $k = \left\lfloor \frac{n-1}{3} \right\rfloor$ is clearly the minimum paired dominating set of $P_n \times P_2$ and S intersecting any maximum independent set of $P_n \square P_2$. Therefore, the set S is a minimum independent transversal paired dominating set of $P_n \square P_2$.

To get the size of the set S, we have $S = \left\{v_i, u_i, where\ i = 3k + 2\ and\ k = 0,1, \dots, \left\lfloor \frac{n-1}{3} \right\rfloor \right\}$. Therefore $|S|$=$\left\lfloor \frac{n-1}{3} \right\rfloor + 1$. Hence $\gamma_{itp}(P_n \square P_2) = 2\left\lfloor \frac{n-1}{3} \right\rfloor + 2$.

Theorem 2.17. *Let C_n be a Cycle with n vertices. Then* $\gamma_{itp}(C_n \square P_2) = 2\left\lfloor \frac{n-1}{3} \right\rfloor + 2$.

Proof. Let $V(C_n \square P_2)$={$v_1, v_2, v_3, \dots, v_n, u_1, u_2, u_3, \dots, u_n$}. Then the product graph $C_n \square P_2$ contains two maximum independent sets of cardinality $2\beta_0(C_n)$.

The set S={$v_l, u_l, v_4, u_4, \dots, v_{3k+1}, u_{3k+1}$} where $k = \left\lfloor \frac{n-1}{3} \right\rfloor$ is clearly a minimum paired dominating set of $C_n \times P_2$ intersecting any maximum independent set of $C_n \square P_2$. Therefore, the set S itself is a minimum independent transversal paired dominating set of $C_n \square P_2$. To get the size of S it is easy to see that S *can* be written as $S = \left\{v_i, u_i, where\ i = 3k + 1\ and\ k = 0,1, \dots, \left\lfloor \frac{n-1}{3} \right\rfloor \right\}$.

That means S Contains $\left\lfloor \frac{n-1}{3} \right\rfloor + 1$ vertex. Hence $\gamma_{itp}(C_n \square P_2) = 2\left\lfloor \frac{n-1}{3} \right\rfloor + 2$.

I. 3. Some Bounds for $\gamma_{itp}(G)$:

We first recall following theorems required for our study:

Theorem 3.1. [11] *Let G be a connected graph with n vertices. Then* 2 < $\gamma_{pd}(G) \leq n$ *and this bounds are sharp.*

Let Φ be the collection of graphs C_3, C_5 and the subdivided star $S(K_{1,n})$. Then we have the following theorem.

Theorem 3.2. [11] *Let G be a connected graph with $n \geq 3$ vertices. Then $\gamma_{pd}(G) \leq n-1$. Equality holds if and only if $G \in \Phi$.*

Theorem 3.3. [8] *Let G be a connected graph with n vertices. Then $\frac{n}{\Delta(G)} \leq \gamma_{itp}(G)$*

We have one of the important remarks here:

Remark 1. *Let S be the minimum independent transversal paired dominating set of G. Then $\langle S \rangle$ only edges in G, it contains no isolated vertices. Hence S is a total dominating set of G. Thus for any graph G, we have $\gamma_t(G) \leq \gamma_{itp}(G)$.*

Theorem 3.4. *Let G be a connected graph with n vertices. Then $\gamma_t(G) \leq \gamma_{pd}(G) \leq \gamma_{itp}(G) \leq n$.*

Proof: Since every independent transversal paired dominating set of G is a paired dominating set of G, we have the second inequality. Further every paired dominating set of G is a total dominating set of G so we get $\gamma_t(G) \leq \gamma_{pd}(G)$.

Theorem 3.5. *Let G be a connected non-complete graph with $n \geq 5$ vertices.*

Then $2 \leq \gamma(G) \leq \gamma_{itp}(G) \leq n-1$.

Proof. Let $V(G) = \{v_1, v_2, \ldots\ldots, v_n\}$. The case n odd is trivial. Assume n is even. Since G is non-complete, $\beta_0(G) \geq 2$. Let S be a γ_{pd}-set of G. Then $|S| \leq n-1$ as the graph G is connected. Further the γ_{pd}-set S intersects the β_0-set of G, for otherwise it will be a contradiction to the order of G. From which it follows that S itself the minimum independent transversal paired dominating set of G and $\gamma_{itp}(G) = |S| \leq n-1$.

Let $\mathcal{F}$ be the collection of complete graphs of even order, any graph of order 4 and mK_2 where $n = 2m$, an even integer.

Theorem 3.6. *Let G be any connected graph with at least 2 vertices. Then $2 \leq \gamma_{itp}(G) \leq n$ and $\gamma_{itp}(G) = n$ if and only if $G \in \mathcal{F}$.*

Proof. Let G be any connected graph, since *V(G)* itself the maximum independent transversal paired dominating set of G and $2 \leq \gamma_{pd}(G)$ always, we get $2 \leq \gamma_{itp}(G) \leq n$. Suppose $\gamma_{itp} = n$. Then if $\beta_0 = 1$ then $G \cong K_{2k}$. Suppose $\beta_0 \geq 2$. On contrary assume $G \notin \mathcal{F}$. Then clearly the order of G must be at least 5. Now if G connected, by Theorem 3.5, $\gamma_{itp}(G) \leq n-1$, a contradiction. Hence if G connected then order of G is at most 4. Clearly it is easy to check that the order of G cannot be less than 4. Hence order of G must be 4. If G cyclic then we have $\gamma_{itp}(G) = 2$, which is not possible. Hence G is acyclic. Further G cannot be a star. Thus G must be a path on 4 vertices, i.e., $G \cong P_4$. If G disconnected, we let $G_1, G_2, \ldots.. G_m$ be the connected components of G.

Since G contains no isolated vertices we must have $m \leq \frac{n}{2}$ and clearly each component has order at least 2. By our assumption $V(G)$ is the minimum independent transversal paired dominating set of G. Suppose if $m < \frac{n}{2}$ then there must be at least one connected component G_k of G with $|V(G_k)| \geq 3$. Hence by Theorem 3.5 there is a vertex $v \in V(G_k)$ such that $V(G_k) \setminus \{v\}$ is a γ_{itp}-set of G_k. Therefore we should have $\gamma_{itp} \leq n-1$, contradicting our assumption. Thus each component of G is of order 2 and there are exactly $\frac{n}{2}$ components. i.e., $G \cong mK_2$ where $n = 2m$.

Theorem 3.7. *For any graph G, we have $\gamma_{pd}(G) \leq \gamma_{itp}(G) \leq \gamma_{pd}(G) + \delta(G)$.*

Proof. First inequality is trivial. Now, suppose v is any vertex of G of degree $\delta(G)$ and S be the minimum paired dominating set of G Then every maximum independent set of G contains a vertex of $N[v]$ so that $S \cup N[u]$ is an independent transversal paired dominating set of G. Further, since S intersects $N[u]$, it follows that $|S \cup N[u]| \leq \gamma_{itp}(G) + \delta(G)$ and hence the second inequality follows.

Remark 2. For any graph $G \in \mathcal{F}$ we have $\gamma_{itp}(G) = n$. In particular if $G \cong K_n$ then- $\gamma_{itp}(K_n) = n$ but $\beta_0(K_n) = 1$. But for the star $K_{1,n-1}$ with $n \geq 3$, we have $\gamma_{itp}(K_{1,n-1}) = 2$ and $\beta_0(K_{1,n-1}) = n$. For the flower graph $f_{m\times n}$, if n odd and m even, then $\gamma_{itp}(f_{m\times n}) = \beta_0 + 1$and for otherwise $\gamma_{itp}(f_{m\times n}) = \beta_0$. Hence for any graph G, there is no relation between its independent transversal paired domination number and the independence number.

II. 4. Maximum degree based vertex addition:

Let G be any graph. Given any graph theoretic parameter of G, the effect of removal of a vertex and an edge on the parameter is of practical importance. As far as our parameter γ_{itp} is concerned, the above mentioned operations may increase, decrease the value of γ_{itp} or it may remain unchanged. Suppose if we consider the operation vertex removal we have the following example:

For the star $K_{1,n-1}(n \geq 4)$, we have $\gamma_{itp}K_{1,n-1})$ = 2, but $\gamma_{itp}(K_{1,n-1} - u)$ = 2 where u is any vertex of the star of degree one. Also $\gamma_{itp}(W_7)$ =3, whereas $\gamma_{itp}(W_7 - v) = 2$, where v is the center vertex of the wheel. Here we consider one more type of operation on G as follows.

Given any two non-adjacent vertices u and v of G, Insert a new vertex w between u and v and the edges uw and vw, if $degu + degv \leq \Delta(G)$,where $\Delta(G)$ denotes the maximum of the degree of G and it will be fixed throughout the operation. We denote the graph obtained from G by applying the above operation by G^*. Clearly this operation may alter the value of $\gamma_{itp}(G)$ or may leave it unchanged. It is easy to observe that by the definition of the operation, $V(G) \leq V(G^*)$ and $E(G) \leq E(G^*)$, equality holds if and only if G contains no non-adjacent vertices.

We have the following easy observations:

Observation 4.1. *$G = G^*$ if and only if G is a complete graph.*

Observation 4.2. *Let G be any graph with n vertices. Then G is always a subgraph of G^*.*

Observation 4.3. *Let G be any graph with n vertices. Then $\Delta(G) = \Delta(G^*)$ and $\delta(G) \leq \delta(G^*)$.*

Figure 2 below shows the maximum degree-vertex addition of the star $K_{1,4}$. Since $\Delta(K_{1,4}) = 4$, we add vertices $u_i$between two successive pairs of pendent vertices of $K_{1,4}$. Note that the line joining u_1 and u_3 is not incident with v_5 and similar holds for the line joining the vertices u_2 and u_4.

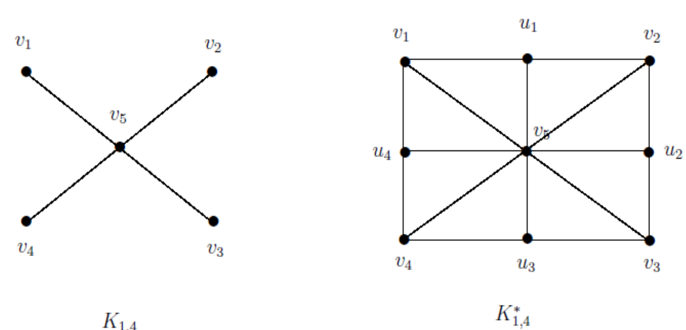

Figure 2:

Consider the following example:

Example 4.1. *Suppose $G \cong P_n$ with $n \neq 4k$ $(k \geq 2)$.Then $G^* \cong C_{n+1}$. Since $\gamma_{itp}(P_n) = \gamma_{itp}(C_n)$, we get $\gamma_{itp}(G) = \gamma_{itp}(G^*)$.*

Example 4.2. *Suppose $G \cong K_{n-1}$ with $n \neq 3$. Then G^* is obtained by inserting the vertex between adjacent pendent edges and joining them by edges, the graph is as shown in figure 2. We have $\gamma_{itp}(K_{1,n-1}) = 2$. We now determine $\gamma_{itp}(K_{1,n-1}^*)$. Since we are adding exactly $n-1$ vertices to $V(K_{1,n-1})$ to obtain $K_{1,n-1}^*$, we have $V(K_{1,n-1}^*) = 2(n-1)$. Further $K_{1,n-1}^*$ contains unique maximum independent set consisting of $n-1$ vertices added and the vertex at the center. Thus*

$\beta_0(K^*_{1,n-1}) = n$. *The* γ_{itp} *set of* $K_{1,n-1}$ *covers all the vertices except the* $n-3$ *vertices added. Hence we must select at least one edge from edges added to obtain the* $\gamma_{itp} - set\ of\ K^*_{1,n-1}$.

Hence $\gamma_{itp}(K_{1,n-1}) < \gamma_{itp}(K^*_{1,n-1})$.

From the above example [5.2], it is proved that the operation above may increase the value of γ_{itp} or remain unchanged. But clearly the operation never decreases the value of γ_{itp}. Therefore, it is possible to partition the class $\mathcal{G}$ of all graphs into two sets $\mathcal{G}^0$ and $\mathcal{G}^+$, where

$$\mathcal{G}_0 = \{G \in \mathcal{G}/\gamma_{itp}(G) = \gamma_{itp}(G^*)\},$$

$$\mathcal{G}^+ = \{G \in \mathcal{G}/\gamma_{itp}(G) > \gamma_{itp}(G^*)\}.$$

Also, from the examples it is evident that the sets $\mathcal{G}^0$ and $\mathcal{G}^+$ are non-empty and the complete graph K_n belongs to none of them. Similarly, one can partition class of graphs into the sets $\mathcal{A}'$, $\mathcal{A}^+$ and $\mathcal{A}^-$ with respect to the operation called edge lifting introduced by T. W. Haynes et al[13]. Now, we can start investigating the properties of these sets.

Partitioning the vertex set $V(G)$ of a graph G into subsets of $V(G)$ having certain property is also one of the direction for the research in graph theory.

For instance, one such partition is domatic partition which is a partition of $V(G)$ into dominating sets. Analogously, we can demand each subset in the partition of $V(G)$ to have the property being independent transversal paired domination instead of domination alone or any other type of domination and call this partition an independent transversal paired domatic partition. Further, since the maximum matching of G is always an independent transversal paired dominating set of G, such partition exists for all graphs except for the complete graphs of odd order so that asking the maximum order of such partition is reasonable; let us call this maximum order as the independent transversal paired domatic number and denote it by $d_{itp}(G)$. Now, begin investigating the parameter d_{itp}.

REFERENCES:

1. J.A. Bondy, U.S.R Murty, *Graph theory with application*, Elsevier science Publishing Co, Sixth printing, 1984

2. G. Chartrand and L. Lesniak, *Graphs and Digraphs* (Fourth edition, CRC Press, Boca Raton, 2005).

3. E. J. Cockayne, R.M. Dawes and S.T. Hedetneimi, *Total domination in graphs*, Networks 10 (1980); 211-219.

4. Eunice Mphako-Banda, *Some Polynomials of Flower Graphs*, International Mathematical Forum, 2, 2007, no. 51, 2511 - 2518.

5. F. Harary, *Graph Theory*, Addison-Wesley, Reading MA, 1969.

6. T. W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs* (Marcel Dekker, New York, 1998).

7. T. W. Haynes, S.T. Hedetniemi and P.J. Slater, *Domination in Graphs: Advanced Topics* (Marcel Dekker, New York, 1998).

8. T. W. Haynes and P.J. Slater, *Paired Domination in Graphs*, Networks, 32(1998) 199-206.

9. S. T. Hedetniemi and R.C. Laskar, *Topics on Domination*, Discrete Math. 86 (1990).

10. Ismail Sahul Hamid, *Independent Transversal Domination in Graphs*, Discussions Mathematicae, Graph Theory 32 (2012) 5-17.

11. V. R. Kulli, *Graph Theory*, Vishwa International Publications, 2000.

12. E. Sampathkumar and H.B. Walikar, *The connected domination number of a graph*, J. Math. Phys. Sci. 13 (1979). 607613.

13. Wyatt J. Desormeaux, T. W. Haynes, M.A. Henning, *Edge lifting and Total Domination in Graphs*, J Comb Optim (2013) 25: 47-59.

14. Saieed Akbari, Aseid Alikhani, Yee-hock Peng *Characterization of graphs using domination polynomials,* Europian journal of Combinatorics 31(2010) 1714-1724.

15. http://mathworld.wolfram.com/SunletGraph.html

16. C. Adiga, A. Bayad, I. Gutman, S. A. Srinivas, The minimum covering energy of a graph, Kragujevac J. Sci. 34 (2012) 39-56.