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The Minimum Paired Dominating Energy of a Graph

Research Article

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Abstract: Let G be any graph. A dominating set D in G is called a paired dominating set if $\langle D\rangle$ contains a perfect matching. The minimum cardinality of a paired-dominating set is called the paired-domination number, denoted by $\gamma_{pd}(G)$. In this paper, we study the minimum paired dominating energy $E_{pd}(G)$, exact value of $E_{pd}(G)$ is calculated for some standard graphs. Further, the bounds are established for $E_{pd}(G)$.

MSC: 05C50, 05C69.

Keywords: Minimum Paired Dominating set, Minimum Paired Dominating matrix, Minimum Paired Dominating Eigenvalues.

1. Introduction

Let $G = (V, E)$ be a graph with $|V| = n$ vertices and $|E| = m$ edges. The concept of energy of a graph G, denoted by $E(G)$ was introduced by I. Gutman in 1978 [3]. Initially, the graph energy concept did not attract any noteworthy attention of mathematicians, but later they did realize its value and worldwide mathematical research of graph energy started. Nowadays, in connection with graph energy, energy-like quantities were considered also for other matrices. In this paper, we are defining a matrix, called the minimum paired dominating matrix denoted by $A_{pd}(G)$ and we study its eigenvalues and the energy. Further, we study the mathematical aspects of the minimum paired dominating energy of a graph. It may be possible that the minimum paired dominating energy which we are considering in this paper have applications in other areas of science such as chemistry and so on. The graphs we are considering are assumed to be finite, simple, undirected having no isolated vertices.

1.1. Definitions

Let G be any graph. A subset M of an edge set E of G is called a matching of G if no two edges in M are incident in G. The two ends of an edge are said to be matched under M. If every vertex of G is matched under M then M is called a perfect matching. M is said to be a maximum matching of G if no subset of G containing M properly is a matching of G. Clearly, every perfect matching is maximum. A subset D of V is called a dominating set of G if every vertex $V - D$ is adjacent to some vertex in D. A dominating set D of a graph G is said to be paired dominating set of G if $\langle D\rangle$ contains atleast one perfect matching. The least cardinality of a paired dominating set in G is called the minimum paired domination number of G, denoted by $\gamma_{pd}(G)$. Any paired dominating set of cardinality $\gamma_{pd}(G)$ is called the $\gamma_{pd}-$set. For more details on the terms used in this paper refer [4].

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Definition 1.1. *The crown graph S_n^0, $(n \geq 2)$ on $2n$ vertices is a graph with vertex set $V = \{u_i, v_j | 1 \leq i, j \leq n\}$ with an edge from u_i to v_j whenever $i \neq j$.*

Definition 1.2. *The double star denoted as $S(n, m)$ with $n \geq m \geq 0$, is the graph consisting of the union of two stars K_n and K_m together with a line joining their centers.*

Definition 1.3. *The cocktail party graph $K_{n\times 2}$ is a graph of order $2n$, with the vertex set $V = \{u_i, v_j | 1 \leq i, j \leq n\}$ and the edge set $E = \{u_iu_j, v_iv_j, u_iv_j, v_iu_j \mid 1 \leq i < j \leq n\}$.*

2. The Minimum Paired Dominating Energy of a Graph

In this section, we define the minimum paired dominating energy of a graph and study some basic properties of the minimum paired dominating energy. The minimum paired dominating matrix of G is an $n-$square matrix defined by $A_{pd}(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if} \quad v_i \;\; and \;\; v_j \;\; are \;\; adjecent, \\ 1 & \text{if} \quad i = j \text{ and } v_i \in D, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_{pd}(G)$ is denoted by $f_n(G, \lambda)$ and is defined by $f_n(G, \lambda) = det(\lambda I - A_{pd}(G))$. The minimum paired dominating eigenvalues of the graph G are the eigenvalues of the matrix $A_{pd}(G)$. We note that these eigenvalues are real numbers since $A_{pd}(G)$ is real and symmetric. So, we can label them in the non-increasing order $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. The minimum paired dominating energy of G is then defined to be the sum of the absolute values of the eigenvalues of $A_{pd}(G)$. In symbols, we write

$$E_{pd}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Example 2.1. *Let G be a tree on 6 vertices as shown in the figure 1, with vertex set $V = \{v_1, v_2, \ldots, v_6\}$ and let $D = \{v_2, v_3\}$ be it's minimum paired dominating set. Then*

$$A_{pd,D}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

So, $f_n(G, \lambda) = \lambda^6 - 2\lambda^5 - 4\lambda^4 + 4\lambda^3 + 4\lambda^2 = 0$. Using *Maple*, we are able to find that $E_{pd}(G) \approx 6.2925$. Now, suppose if we choose another paired dominating set $D' = \{v_1, v_2, v_3, v_4\}$. Then

$$A_{pd,D'}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly, $f_n(G,\lambda) = \lambda^6 - 4\lambda^5 + 1\lambda^4 + 8\lambda^3 - 4\lambda^2 + 2\lambda + 1 = 0$. Using *Maple*, we are able to find that $E_{pd,D'}(G) \approx 7.5581$. Therefore, it is clear from the above example that the minimum paired dominating energy of a graph G depends on the minimum paired dominating set we choose. Hence, this energy is not the graph invariant.

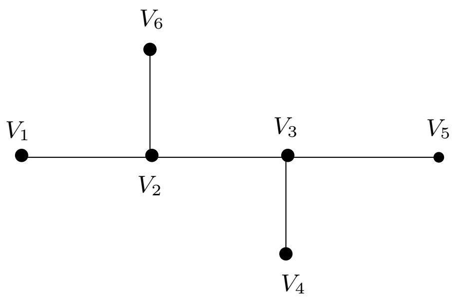

Fig. 1

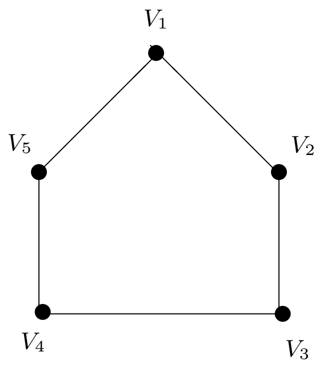

Fig. 2

Proof of the following theorem is easy and straightforward.

Theorem 2.2. *Let G be a graph with an edge set E and let γ_{pd} be the paired domination number of G. If $f_n(G,\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$. Then*

(1). $a_0 = 1$.

(2). $a_1 = -\gamma_{pd}$.

(3). $a_2 = \frac{\gamma_{pd}(\gamma_{pd}-1)}{2} - |E|$.

We note that sum of the squares of the eigenvalues of $A_{pd}(G)$ is the trace of $(A_{pd}(G))^2$. So we have the following theorem.

Theorem 2.3. *Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ denotes the eigenvalues of $A_{pd}(G)$. Then* $\sum_{i=1}^{n} {\lambda_i}^2 = 2m + \gamma_{pd}(G)$.

3. The Minimum Paired Dominating Energy of Some Standard Graphs

In this section, we calculate the minimum paired dominating energy of some standard graphs.

Definition 3.1. *Suppose $\lambda_1, \lambda_2, \cdots \lambda_n$ are the distinct eigenvalues of G with the multiplicities $m_1, m_2, \cdots m_n$ respectively, then the minimum paired dominating spectrum of the graph G will be written as*

$$MPDSpec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}.$$

Theorem 3.2. *For $n \geq 3$, the minimum paired dominating energy of a complete graph K_n is* $(n-3) + \sqrt{n^2 - 2n + 9}$.

Proof. Let K_n be the complete graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Clearly, $D = \{v_1, v_2\}$ is a minimum paired dominating set of K_n. Then

$$A_{pd}(K_n) = \begin{pmatrix} 1 & 1 & 0 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \dots & 1 & 1 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 1 & 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}.$$

So, $f_n(G, \lambda) = \lambda(\lambda+1)^{n-3}(\lambda^2 - (n-1)\lambda - 2)$ and so we obtain that

$$MPDSpec(K_n) = \begin{pmatrix} 0 & -1 & \frac{(n-1)+\sqrt{(n^2-2n+9)}}{2} & \frac{(n-1)-\sqrt{(n^2-2n+9)}}{2} \\ 1 & n-3 & 1 & 1 \end{pmatrix}.$$

Therefore, $E_{pd}(K_n) = (n-3) + \sqrt{n^2-2n+9}$. □

Theorem 3.3. *For $n \geq 3$, the minimum paired dominating energy of a crown graph S_n^0 is equal to $(2n-4)+\sqrt{n^2+2n-7}+\sqrt{n^2-2n+9}$.*

Proof. Let S_n^0 be a crown graph and let $V(S_n^0) = \{u_1, u_2, u_3, \dots, u_n, v_1, v_2, v_3, \dots, v_n\}$. Then the minimum paired dominating set is given by $D = \{u_1, u_2, v_1, v_2\}$. Then

$$A_{pd}(S_n^0) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1 \\ 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 0 & 1 & 1 & 0 & \dots & 1 \\ . & . & . & \dots & . & . & . & . & \dots & 1 \\ . & . & . & \dots & . & . & . & . & \dots & 1 \\ . & . & . & \dots & . & . & . & . & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 1 & 0 & 1 & 0 & \dots & 0 \\ . & . & . & \dots & . & . & . & . & \dots & 0 \\ . & . & . & \dots & . & . & . & . & \dots & 0 \\ . & . & . & \dots & . & . & . & . & \dots & 0 \\ 1 & 1 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

So, $f_n(S_n^0, \lambda) = \lambda(\lambda-2)(\lambda-1)^{(n-3)}(\lambda+1)^{n-3}(\lambda^2+(n-3)\lambda-(2n-4))$. Hence,

$$MPDSpec(S_n^0) = \begin{pmatrix} 0 & 2 & 1 & -1 & \frac{(n-1)+\sqrt{(n^2-2n+9)}}{2} & \frac{(n-1)-\sqrt{(n^2-2n+9)}}{2} & \frac{(3-n)+\sqrt{(n^2+2n-7)}}{2} & \frac{(3-n)-\sqrt{(n^2+2n-7)}}{2} \\ 1 & 1 & n-3 & n-3 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Therefore $E_{pd}(S_n^0) = (2n-4) + \sqrt{n^2+2n-7} + \sqrt{n^2-2n+9}$. □

Theorem 3.4. *For $n \geq 2$, the minimum paired dominating energy of a star $K_{1,n-1}$ is at most $2+2\sqrt{(n-2)}$. Equality holds if and only if $n = 2$.*

Proof. Let $K_{1,n-1}$ be a star with the vertex set $V = \{v_1, v_2, \dots, v_n\}$ having the vertex v_n at the center. The minimum paired dominating set is $D = \{v_1, v_n\}$. Then

$$A_{pd}(K_{1,n-1}) = \begin{pmatrix} 1 & 0 & 0 & ... & 0 & 1 \\ 0 & 0 & 0 & ... & 0 & 1 \\ 0 & 0 & 0 & ... & 0 & 1 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 0 & 1 \\ 1 & 1 & 1 & ... & 1 & 1 \end{pmatrix}.$$

So, $f_n(S_n^0, \lambda) = \lambda^{n-3}(\lambda^3 - 2\lambda^2 - (n-2)\lambda) + (n-2)$. By using simple analysis, we obtain that

$$\begin{aligned} f_n(K_{1,n-1}, \lambda) &= \lambda^{n-3}[\lambda^3 - 2\lambda^2 - (n-2)\lambda) + (n-2)] \\ &= \lambda^{n-3}[\lambda^3 - 2\lambda^2 - (n-2)\lambda) + 2(n-2) - (n-2)] \\ &\leq \lambda^{n-3}[\lambda^3 - 2\lambda^2 - (n-2)\lambda) + 2(n-2)] \\ &= \lambda^{n-3}[(\lambda - 2)(\lambda^2 - (n-2))]. \end{aligned}$$

Hence, $MPDSpec(K_{1,n-1}) \approx \begin{pmatrix} 0 & 2 & \sqrt{(n-2)} & -\sqrt{(n-2)} \\ n-3 & 1 & 1 & 1 \end{pmatrix}$. Therefore, $E_{pd}(K_{1,n-1}) \leq 2 + 2\sqrt{(n-2)}$. □

Corollary 3.5. *The minimum paired dominating energy of $L(K_{1,n-1})$ is equal to $(n-4) + \sqrt{(n^2 - 4n + 12)}$.*

Proof. We first note that The line graph of $K_{1,n-1}$ is the complete graph K_{n-1}. Now, it follows easily that $E_{pd}(L(K_{1,n-1})) = E_{pd}(K_{n-1})$. Hence from Theorem 3.1, we obtain, $E_{pd}(K_{1,n-1}) = (n-4) + \sqrt{(n^2 - 4n + 12)}$. □

Theorem 3.6. *For $n \geq m \geq 2$, the minimum paired dominating energy of a double star $S(n,m)$ is greater than or equal to $2\sqrt{(m+n)-1}$. In particular for m=n with $n \geq 5$ the energy of a double star is equal to $2(\sqrt{n-1} + \sqrt{n})$.*

Proof. Suppose $S(n,m)$, with $n, m \geq 2$ is a double star having the vertex set $V = \{u_1, u_2, u_3, \dots, u_n, v_1, v_2, v_3, \dots v_m\}$ with u_n and v_m as the vertices at the center. Clearly minimum paired dominating set is $D = \{u_n, v_m\}$. Then

$$A_{pd}(S(n,m)) = \begin{pmatrix} 0 & 0 & 0 & ... & 1 & 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 1 & 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 1 & 0 & 0 & 0 & ... & 0 \\ . & . & . & ... & . & . & . & . & ... & 0 \\ . & . & . & ... & . & . & . & . & ... & 0 \\ . & . & . & ... & . & . & . & . & ... & 0 \\ 1 & 1 & 1 & ... & 1 & 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 & ... & 1 \\ . & . & . & ... & . & . & . & . & ... & 1 \\ . & . & . & ... & . & . & . & . & ... & 1 \\ . & . & . & ... & . & . & . & . & ... & 1 \\ 0 & 0 & 0 & ... & 1 & 1 & 1 & 1 & ... & 1 \end{pmatrix}.$$

So, $f_n(S(n,m),\lambda) = \lambda^{m+n-4}(\lambda^4 - 2\lambda^3 - (m+n-2)\lambda^2 + (m+n-2)\lambda + (m-1)(n-1))$. By using simple analysis,

$$\begin{aligned}
f_{n+m}(S(n,m),\lambda) &= \lambda^{m+n-4}(\lambda^4 - 2\lambda^3 - (m+n-2)\lambda^2 + (m+n-2)\lambda + (m-1)(n-1)) \\
&\geq \lambda^{m+n-4}(\lambda^4 - 2\lambda^3 - (m+n-2)\lambda^2 + (m+n-2)\lambda) \\
&\geq \lambda^{m+n-3}(\lambda^3 - 2\lambda^2 - (m+n-2)\lambda) \\
&= \lambda^{m+n-2}(\lambda^2 - 2\lambda - (m+n-2)).
\end{aligned}$$

Hence, $MPDSpec(S(n,m)) = \begin{pmatrix} 0 & 1+\sqrt{(m+n)-1} & 1-\sqrt{(m+n)-1} \\ m+n-2 & 1 & 1 \end{pmatrix}$

Therefore, we get $E_{pd}(S(n,m)) \geq 2\sqrt{(m+n)-1}$. In particular, for m=n, we have

$$\begin{aligned}
f_{2n}(S(n,n),\lambda) &= \lambda^{2n-4}(\lambda^4 - 2\lambda^3 - 2(n-1)\lambda^2 + 2(n-1)\lambda + (n-1)^2) \\
&= \lambda^{2n-4}(\lambda^2 - (n-1))(\lambda^2 - 2\lambda - (n-1))
\end{aligned}$$

Hence, $MPDSpec(S(n,n)) = \begin{pmatrix} 0 & -\sqrt{n-1} & \sqrt{n-1} & 1+\sqrt{n} & 1-\sqrt{n} \\ 2n-4 & 1 & 1 & 1 & 1 \end{pmatrix}$. Therefore, we get $E_{pd}(S(n,n)) = 2(\sqrt{n-1} + \sqrt{n})$. □

Theorem 3.7. *For $n \geq 3$, the minimum paired dominating energy of a cocktail party graph $K_{n\times 2}$ is at least $2n-6+\sqrt{5}+\sqrt{4n^2-4n+13}$.*

Proof. Let $K_{n\times 2}$ be the cocktail party graph with the vertex set $V = \{u_i, v_i | 1 \leq i \leq n\}$ and minimum paired dominating set be $D = \{u_1, v_2\}$. Then

$$A_{pd}(K_{n\times 2}) = \begin{pmatrix}
1 & 1 & 1 & ... & 1 & 0 & 1 & 1 & ... & 1 \\
1 & 0 & 1 & ... & 1 & 1 & 0 & 1 & ... & 1 \\
1 & 1 & 0 & ... & 1 & 1 & 1 & 0 & ... & 1 \\
. & . & . & ... & . & . & . & . & ... & 1 \\
. & . & . & ... & . & . & . & . & ... & 1 \\
. & . & . & ... & . & . & . & . & ... & 1 \\
1 & 1 & 1 & ... & 0 & 1 & 1 & 1 & ... & 0 \\
0 & 1 & 1 & ... & 1 & 0 & 1 & 1 & ... & 1 \\
1 & 0 & 1 & ... & 1 & 1 & 1 & 1 & ... & 1 \\
. & . & . & ... & . & . & . & . & ... & 1 \\
. & . & . & ... & . & . & . & . & ... & 1 \\
. & . & . & ... & . & . & . & . & ... & 1 \\
1 & 1 & 1 & ... & 0 & 1 & 1 & 1 & ... & 0
\end{pmatrix}.$$

So, $f_n(K_{n\times 2},\lambda) = \lambda^{n-2}(\lambda+2)^{n-3}(\lambda^2+\lambda-1)(\lambda^3 - (2n-3)\lambda^2 - (2n+1)\lambda + (2n-2))$.

By using analysis,

$$\begin{aligned}
f_n(K_{n\times 2},\lambda) &= \lambda^{n-2}(\lambda+2)^{n-3}(\lambda^2+\lambda-1)[\lambda^3 - (2n-3)\lambda^2 - (2n+1)\lambda + (2n-2)] \\
&\geq \lambda^{n-2}(\lambda+2)^{n-3}(\lambda^2+\lambda-1)[\lambda^3 - (2n-3)\lambda^2 - (2n+1)\lambda] \\
&= \lambda^{n-1}(\lambda+2)^{n-3}(\lambda^2+\lambda-1)[\lambda^2 - (2n-3)\lambda - (2n+1)]
\end{aligned}$$

Hence, $MPDSpec(K_{n\times 2}) \approx \begin{pmatrix} 0 & -2 & \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} & \frac{(2n-3)+\sqrt{4n^2-4n+13}}{2} & \frac{(2n-3)-\sqrt{4n^2-4n+13}}{2} \\ n-1 & n-3 & 1 & 1 & 1 & 1 \end{pmatrix}$

Therefore, $E_{pd}(K_{n\times 2)} \geq 2n-6+\sqrt{5}+\sqrt{4n^2-4n+13}$. □

Theorem 3.1. *For $n \geq 3$, the minimum paired dominating energy of a complete bipartite graph $K_{n,n}$ is $(2n-4)+\sqrt{n^2-2n+5}+\sqrt{n^2+2n+3}$.*

Proof. Let $K_{m,n}$ be a complete bipartite graph and let $V(K_{m,n}) = \{u_i, v_j | 1 \leq i \leq m, 1 \leq j \leq n\}$. Then the minimum paired dominating set is $D= \{u_1, v_1\}$. Hence

$$A_{pd}(K_{m,n}) = \begin{pmatrix} 1 & 0 & 0 & ... & 0 & 1 & 1 & 1 & ... & 1 \\ 0 & 0 & 0 & ... & 0 & 1 & 1 & 1 & ... & 1 \\ 0 & 0 & 0 & ... & 0 & 1 & 1 & 1 & ... & 1 \\ . & . & . & ... & . & . & . & . & ... & 1 \\ . & . & . & ... & . & . & . & . & ... & 1 \\ . & . & . & ... & . & . & . & . & ... & 1 \\ 0 & 0 & 0 & ... & 0 & 1 & 1 & 1 & ... & 1 \\ 1 & 1 & 1 & ... & 1 & 1 & 0 & 0 & ... & 0 \\ 1 & 1 & 1 & ... & 1 & 0 & 0 & 0 & ... & 0 \\ . & . & . & ... & . & . & . & . & ... & 0 \\ . & . & . & ... & . & . & . & . & ... & 0 \\ . & . & . & ... & . & . & . & . & ... & 0 \\ 1 & 1 & 1 & ... & 1 & 0 & 0 & 0 & ... & 0 \end{pmatrix}.$$

So, $f_n(K_{m,n}, \lambda) = \lambda^{m+n-4}[\lambda^4 - 2\lambda^3 - (mn-1)\lambda^2 + (2mn-m-n)\lambda - (m-1)(n-1)]$.

In particular, for $m = n$ we have,

$$f_{2n}(K_{n,n}, \lambda) = (\lambda-1)^{n-2}(\lambda+1)^{n-2}(\lambda^2+(n-3)\lambda-(2n-3))(\lambda^2-(n-1)\lambda-1)).$$

Hence, $MPDSpec(K_{n,n})=\begin{pmatrix} -1 & 1 & \frac{(3-n)+\sqrt{n^2+2n-3}}{2} & \frac{(3-n)-\sqrt{n^2+2n-3}}{2} & \frac{(n-1)-\sqrt{n^2-2n+5}}{2} & \frac{(n-1)-\sqrt{n^2-2n+5}}{2} \\ n-2 & n-2 & 1 & 1 & 1 & 1 \end{pmatrix}$.

Therefore, we get $E_{pd}(K_{n,n}) = (2n-4)+\sqrt{n^2+2n-3}+\sqrt{n^2-2n+5}$. □

4. Upper and Lower Bounds

Theorem 4.1. *Let G be any graph. Then $E_{pd}(G) \leq \sqrt{n(2m+\gamma_{pd})}$.*

Proof. Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of $A_{pd}(G)$ arranged in the non-increasing order. By Cauchy Schwarz inequality, we have

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big)$$

If we take, $a_i = 1$, $b_i =| \lambda_i |$ then

$$\Big(\sum_{i=1}^{n} | \lambda_i |\Big)^2 \leq \Big(\sum_{i=1}^{n} 1\Big)\Big(\sum_{i=1}^{n} \lambda_i^2\Big)$$

Therefore by theorem 2.2, we obtain,

$$(E_{pd}(G))^2 \leq n\Big(\sum_{i=1}^{n} \lambda_i^2\Big) = n(2m+\gamma_{pd})$$

$$i.e., \; E_{pd}(G) \leq \sqrt{n(2m+\gamma_{pd})}.$$

Theorem 4.2. *For any graph G, we have* $\sqrt{m+\frac{\gamma_{pd}(\gamma_{pd}+1)}{2}} \leq E_{pd}(G) \leq 2\sqrt{m(m+\frac{\gamma_{pd}}{2})}$.

Proof. We first assume that the graph G, we are considering has no isolated vertices. Then, from the definition of minimum paired dominating energy we have,

$$(E_{pd}(G))^2 = \sum_{i=1}^{n} |\lambda_i|^2 + 2\sum_{i<j} |\lambda_i \lambda_j|.$$

From theorem 2.2, we get

$$(E_{pd}(G))^2 = (2m+\gamma_{pd}) + 2\sum_{i<j} |\lambda_i \lambda_j|.$$

Also, we have from theorem 2.1

$$\sum_{i<j} |\lambda_i||\lambda_j| \geq \left|\sum_{i<j} \lambda_i \lambda_j\right| = \frac{\gamma_{pd}(\gamma_{pd}-1)}{2} - m$$

Consequently

$$(E_{pd}(G))^2 \geq 2m + \gamma_{pd} + \tfrac{\gamma_{pd}(\gamma_{pd}-1)}{2} - m$$

$$i.e., E_{pd}(G) \geq \sqrt{m+\tfrac{\gamma_{pd}(\gamma_{pd}+1)}{2}}.$$

Further, the maximum number of vertices of such graphs is 2m, From Theorem 4.1, we have

$$E_{pd}(G) \leq 2\sqrt{m\left(m+\frac{\gamma_{pd}}{2}\right)}$$

Since, γ_{pd} is an even integer always, $\frac{\gamma_{pd}}{2}$ will be an integer. So Combining we get,

$$\sqrt{m+\tfrac{\gamma_{pd}(\gamma_{pd}+1)}{2}} \leq E_{pd}(G) \leq 2\sqrt{m(m+\tfrac{\gamma_{pd}}{2})}.$$

□

Theorem 4.3. *(Lower bound) Let G be any graph. Then* $E_{pd}(G) \geq \sqrt{2m+\gamma_{pd}}$.

Proof. If $a_1, a_2, a_3, \ldots, a_n$ are positive real numbers. Then $\left(\sum_{i=1}^{n} |a_i|\right)^2 \geq \left(\sum_{i=1}^{n} a_i^2\right)$.
For each i, taking $a_i = |\lambda_i|$, we obtain,

$$\Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2 \geq \Big(\sum_{i=1}^{n} \lambda_i^2\Big) \tag{1}$$

Then,

$$(E_{pd}(G))^2 \geq \Big(\sum_{i=1}^{n} \lambda_i^2\Big) = 2m + \gamma_{pd} \tag{2}$$

Therefore, we obtain that $E_{pd}(G) \geq \sqrt{2m+\gamma_{pd}}$. □

Theorem 4.4. *Let G be any graph having n vertices and m edges. Then* $E_{pd}(G) \geq \sqrt{2m+\gamma_{pd}+n(n-1)|A_{pd}(G)|^{\frac{2}{n}}}$.

Proof. By definition we have,

$$\begin{aligned}(E_{pd}(G))^2 &= \Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2.\\ &= \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i\neq j} |\lambda_i||\lambda_j|.\end{aligned}$$

Now, by using arithmetic and geometric mean inequality, we get

$$\begin{aligned}\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| &\geq \Big[\prod_{i\neq j}|\lambda_i||\lambda_j|\Big]^{\frac{1}{n(n-1)}}\\ &= \Big[\prod_{i=1}^{n}|\lambda_i|^{2(n-1)}\Big]^{\frac{1}{n(n-1)}}\\ &= \Big|\prod_{i=1}^{n}\lambda_i\Big|^{\frac{2}{n}}\\ &= (detA_{pd}(G))^{\frac{2}{n}}\end{aligned}$$

Hence,

$$\sum_{i\neq j}|\lambda_i||\lambda_j| \geq n(n-1)(detA_{pd}(G))^{\frac{2}{n}}$$

Now consider

$$\begin{aligned}[E_{pd}(G)]^2 &= \sum_{i=1}^{n}|\lambda_i|^2+\sum_{i\neq j}|\lambda_i||\lambda_j|.\\ &\geq \sum_{i=1}^{n}|\lambda_i|^2+n(n-1)(detA_{pd}(G))^{\frac{2}{n}}\\ i.e.,\ E_{pd}(G) &\geq \sqrt{2m+\gamma_{pd}+n(n-1)|A_{pd}(G)|^{\frac{2}{n}}}.\end{aligned}$$

□

Theorem 4.5. *If G is $r-$regular graph with n vertices and m edges and let λ_n denotes the largest eigenvalue of G. Then $E_{pd}(G)\leq|\lambda_n|+\sqrt{(n-1)(nr+\gamma_{pd}-(|\lambda_n|)^2}$.*

Proof. Let G be an $r-$regular graph with n vertices and m edges. Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the eigenvalues of G and let λ_n denote the largest eigenvalue. Then taking $a_i=1$, $b_i=|\lambda_i|$ in Cauchy's-Schwarz inequality, we get

$$\Big(\sum_{i=1}^{n-1}|\lambda_i|\Big)^2\leq(n-1)\Big(\sum_{i=1}^{n-1}\lambda_i^2\Big) \tag{3}$$

and

$$E_{pd}(G)-|\lambda_n|=\sum_{i=1}^{n-1}|\lambda_i| \tag{4}$$

Also,

$$2m+\gamma_{pd}=\sum_{i=1}^{n}|\lambda_i|^2=\sum_{i=1}^{n-1}|\lambda_i|^2+|\lambda_n|^2$$

Hence

$$2m+\gamma_{pd}-|\lambda_n|^2=\Big(\sum_{i=1}^{n-1}|\lambda_i|^2\Big) \tag{5}$$

Now, from $(1),(2)$ and (3) we have

$$(E_{pd}(G)-|\lambda_n|)^2\leq(n-1)(2m+\gamma_{pd}-|\lambda_n|^2)$$

Since $2m=nr$, we obtain that $E_{pd}(G)\leq|\lambda_n|+\sqrt{(n-1)(nr+\gamma_{pd}-(|\lambda_n|)^2)}$. □

Definition 4.1. *For any two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, the union of G_1 and G_2 is the graph $G=G_1\cup G_2$ having the vertex set $V=V_1\cup V_2$ and the edge set $E=E_1\cup E_2$.*

The union of two graphs which are not disjoint is also defined in the same way. By using the simple linear algebra, we have the following proposition.

Proposition 4.2. *Let G_1 and G_2 are any two graphs. Then, we have $E_{pd}(G_1\cup G_2)=E_{pd}(G_1)+E_{pd}(G_2)$.*

4.1. Some properties of $E_{pd}(G)$

Here, we look for the graph G whose minimum paired dominating energy is equal to the number of vertices of G. In fact if $G \cong \frac{n}{2}K_2$ then $E_{pd}(G) = n$. clearly we must have n is an even integer, i.e $n = 2m$ for some integer m. Suppose $G \cong \frac{n}{2}K_2$ having the vertex set $V = \{v_1, v_2, v_3, ...v_{2m}\}$. Clearly V itself is a minimum paired dominating set of G. Then,

$$A_{pd}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & ... & 0 & 0 \\ 1 & 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & 1 & ... & 0 & 0 \\ 0 & 0 & 1 & 1 & ... & 0 & 0 \\ . & . & . & ... & . & . & . \\ . & . & . & ... & . & . & . \\ . & . & . & ... & . & . & . \\ 0 & 0 & 0 & ... & 0 & 1 & 1 \\ 0 & 0 & 0 & ... & 0 & 1 & 1 \end{pmatrix}.$$

The characteristic equation of $A_{pd}(G)$ is $\lambda^m(\lambda - 2)^m$. Hence, $MPDSpec(G) = \begin{pmatrix} 0 & 2 \\ m & m \end{pmatrix}$.
Therefore, $E_{pd}(G) = |0|.m + |2|.m = 2m$. i.e., $E_{pd}(G) = n$, the number of vertices in G. Thus, every even positive integer can be looked as a paired dominating energy of some graph. R.B.Bapat and Pati [2] showed that, if the energy of a graph is rational then it must be an even number. Analogously we have the following theorem.

Theorem 4.6. *Let G be a graph with a minimum paired dominating set D. If the minimum paired dominating energy $E_{pd}(G)$ of G is a rational number. Then*

$$E_{pd}(G) \equiv \gamma_{pd}(G)(mod\ 2).$$

Recently C.Adiga et al studied the minimum covering energy of a graph, for more details we refer [1]. It is proved that each positive integer $2p-1(\geq 3)$ can be looked as the minimum covering energy of a star graph K_{1,p^2-p}. Given any graph G and a dominating set D we have minimum dominating energy introduced and studied by Rajesh Kannan et al [5]. For any positive integer n, consider a graph G on n vertices which is totally disconnected having the vertex set $V = \{v_1, v_2, v_3, ...v_n\}$, clearly V itself is a minimum dominating set of G. Then $A_D(G)$ is an identity matrix of order n, whose characteristic polynomial is $(\lambda - 1)^n$. Hence the minimum dominating eigenvalue of G is 1 with the multiplicity n. Hence the minimum dominating energy of G will be n. Therefore, it follows from the above that every positive integer can be looked as the dominating energy of a graph. Any two graphs non-isomorphic G and H for which $E_{pd}(G) = E_{pd}(H)$ are called equi-paired dominating energetic graphs.

4.2. Hyper-paired Dominating Energetic Graph

I. Gutman conjectured that among all graphs on n vertices K_n, the complete graph has the maximum energy. But later it is proved that there are graphs having energy greater than the energy of a complete graph such graphs are called hyperenergetic graphs for more details on hyper-energetic graphs we refer to [3].
The graphs whose energy less than the energy of a complete graph are called non-hyper-energetic graphs. Here, we are interested in graphs having the minimum paired dominating energy greater than that of the complete graph and we refer them as hyper-paired dominating energetic graphs. From Theorem 3.1 we have, for $n \geq 3$, the minimum paired dominating energy of a complete graph is $(n-3) + \sqrt{n^2 - 2n + 9}$.

In this section we investigate for the graph G on n vertices having the minimum paired dominating energy greater than the minimum paired dominating energy of a complete graph. In fact we note that the energy of a cocktail party graph is at least $2n-6+\sqrt{5}+\sqrt{4n^2-4n+13}$. clearly, for $n \geq 3$, we have $4n^2-4n+13 > n^2-2n+9$. Hence,

$$\begin{aligned} E_{pd}(K_n) &= (n-3)+\sqrt{n^2-2n+9} \\ &\leq 2(n-3)+\sqrt{n^2-2n+9} \\ &\leq 2n-6+\sqrt{4n^2-4n+13}+\sqrt{5} \\ &\leq E_{pd}(K_{n\times 2}). \end{aligned}$$

Thus, Cockatail party graph is an hyper-paired dominating energetic graph. The line graph $L(K_n)$ of a complete graph is also hyper paired-hyper dominating energetic graph. The graphs having the minimum paired dominating energy less than that of the complete graph are referred as non hyper-paired dominating energetic graphs, star graph and double-star graphs are examples for the non hyper-paired dominating energetic graphs.

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