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The Minimum Boundary Dominating Energy of a Graph

Research Article

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Abstract: For a graph G, a subset B of $V(G)$ is called a boundary dominating set if every vertex of $V(G) - S$ is vertex boundary dominated by some vertex of S. The boundary domination number $\gamma_b(G)$ of G is the minimum cardinality of minimum boundary dominating set in G. In this paper we introduce the minimum boundary dominating energy $E_B(G)$ of a graph G and computed minimum boundary dominating energies of some standard graphs. Upper and lower bounds for $E_B(G)$ are established.

MSC: 05C69 05C38

Keywords: Boundary dominating set, Minimum Boundary Domination Matrix, Minimum Boundary Dominating Eigenvalues, Minimum Boundary Dominating Energy of a Graph.

1. Introduction

In this paper, by a graph $G(V, E)$ we mean a simple graph, that is nonempty, finite, having no loops, no multiple and directed edges. Let n and m be the number of vertices and edges, respectively, of G. The degree of a vertex v in a graph G, denoted by $deg(v)$, is the number of vertices adjacent to v. For any vertex v of a graph G, the open neighborhood of v is the set $N(v) = \{u \in V : uv \in E(G)\}$. For a subset $S \subseteq V(G)$ the degree of a vertex $v \in V(G)$ with respect to a subset S is $deg_S(v) = |N(v) \cap S|$. For graph theoretic terminology we refer to Harary book [8].

The distance between two vertices u and v is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u. A vertex v is called an eccentric vertex of u if $e(u) = d(u, v)$. A vertex v is an eccentric vertex of G if v is an eccentric vertex of some vertex of G. Consequently if v is an eccentric vertex of u and w is a neighbor of v, then $d(u, w) \leq d(u, v)$. A vertex v may have this property, however, without being an eccentric vertex of u.

Let G be a simple graph $G = (V, E)$ with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. For $i \neq j$, a vertex v_i is a boundary vertex of v_j if $d(v_j, v_t) \leq d(v_j, v_i)$ for all $v_t \in N(v_i)$ [4].

A vertex v is called a boundary neighbor of u if v is a nearest boundary of u. If $u \in V$, then the boundary neighbourhood of u denoted by $N_b(u)$ is defined as $N_b(u) = \{v \in V : d(u, w) \leq d(u, v)$ for all $w \in N(u)\}$. The cardinality of $N_b(u)$ is denoted by $deg_b(u)$ in G. The maximum and minimum boundary degree of a vertex in G are denoted respectively by $\Delta_b(G)$ and

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$\delta_b(G)$. That is $\Delta_b(G) = \max_{u\in V} |N_b(u)|$, $\delta_b(G) = \min_{u\in V} |N_b(u)|$.

A vertex u boundary dominate a vertex v if v is a boundary neighbor of u. KM.Kathiresan, G. Marimuthu and M. Sivanandha Saraswathy introduced the concept of Boundary domination in graphs. A subset B of $V(G)$ is called a boundary dominating set if every vertex of $V - B$ is boundary dominated by some vertex of B. The minimum taken over all boundary dominating sets of a graph G is called the boundary domination number of G and is denoted by $\gamma_b(G)$.[10].

The concept energy of a graph introduced by I. Gutman [6] in the year 1978. Let $A(G) = (a_{ij})$ be the adjacency matrix of G. The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of a matrix $A(G)$, assumed in nonincreasing order, are the eigenvalues of the graph G. Let $\lambda_1, \lambda_2, ..., \lambda_r$ for $r \leq n$ be the distinct eigenvalues of G with multiplicity $m_1, m_2, ..., m_r$, respectively, the multiset of eigenvalues of $A(G)$ is called the spectrum of G and denoted by

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}$$

As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G, i.e. $E(G) = \sum_{i=1}^{n} |\lambda_i|$. For more details on the mathematical aspects of the theory of graph energy we refer to [2], [7], [12]. Recently C. Adiga et al. [1] defined the minimum covering energy, $E_C(G)$ of a graph which depends on its particular minimum cover C. Motivated by this paper, we introduce minimum boundary dominating energy, denoted by $E_B(G)$, of a graph G, and computed minimum boundary dominating energies of some standard graphs. Upper and lower bounds for $E_B(G)$ are established. It is possible that the boundary dominating energy that we are considering in this paper may be have some applications in chemistry as well as in other areas.

2. The Minimum Boundary Dominating Energy of Graphs

Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set E. A subset B of $V(G)$ is called a boundary dominating set if every vertex of $V - B$ is boundary dominated by some vertex of B. The boundary domination number $\gamma_b(G)$ of G is the minimum cardinality of a boundary dominating set. Any boundary dominating set with minimum cardinality is called a MBD set. Let B be a MBD set of a graph G. The MBD matrix of G is the $n \times n$ matrix defined by $A_B(G) = a_{ij}$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_j \in N_b(v_i), \\ 1 & \text{if } i = j \text{ and } v_i \in B, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_B(G)$ is denoted by

$$f_n(G, \lambda) = \det(\lambda I - A_B(G))$$

The MBD eigenvalues of the graph G are the eigenvalues of $A_B(G)$. Since $A_B(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The MBD energy of G is defined as $E_B(G) = \sum_{i=1}^{n} |\lambda_i|$. We first compute the MBD energy of a graph in Figure 1.

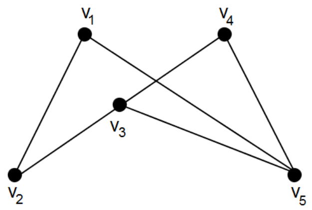

Figure 1. G

Example 2.1. *Let G be a graph in fig 1 with vertices set* $\{v_1, v_2, v_3, v_4, v_5\}$ *and let its MBD set be* $B_1 = \{v_1, v_5\}$*. Then*

$$A_{B_1}(G) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Characteristic equation is $f_n(G, \lambda) = \lambda^5 - 2\lambda^4 - 3\lambda^3 + 4\lambda^2 + 2\lambda - 1 = 0$*. Hence, the MBD eigenvalues are* $\lambda_1 \approx 2.4498, \lambda_2 \approx -1.3354, \lambda_3 \approx 1.2607, \lambda_4 \approx -0.7145, \lambda_5 \approx 0.3393$*. Therefore the MBD energy of G is*

$$E_{B_1}(G) \approx 6.0997.$$

If we take another MBD set in G, namely $B_2 = \{v_2, v_3\}$*, then*

$$A_{B_2}(G) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Characteristic equation is $f_n(G, \lambda) = \lambda^5 - 2\lambda^4 - 3\lambda^3 + 5\lambda^2 + 2\lambda - 1 = 0$*. Hence,the MBD eigenvalues are* $\lambda_1 \approx 2.1701, \lambda_2 \approx 1.6180, \lambda_3 \approx -1.4812, \lambda_4 \approx -0.6180, \lambda_5 \approx 0.3111$*. Therefore the MBD energy of G is*

$$E_{B_2}(G) \approx 6.1984.$$

This example illustrates the fact that the MBD energy of a graph G depends on the choice of the MBD set. i.e. the MBD energy is not a graph invariant.

3. Some Properties of MBD Energy of Graphs

In this section, we introduce some properties of characteristic polynomials of MBD matrix and some properties of minimum boundary dominating eigenvalues of a graph G.

Theorem 3.1. *Let G be a graph of order and size n and m, respectively. Let*

$$f_n(G, \lambda) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2}.. + c_{n-1}\lambda^2 + c_n$$

be the characteristic polynomials of MBD matrix of a graph G. Then

(1). $c_0 = 1$.

(2). $c_1 = -|B|$.

(3). $c_2 = \begin{pmatrix} |B| \\ 2 \end{pmatrix}$ - m.

Proof.

(1). From the definition of $f_n(G, \lambda)$.

(2). Since the sum of diagonal elements of $A_B(G)$ is equal to $|B|$, where B is a MBD set a graph G. The sum of determinants of all 1×1 principal submatrices of $A_B(G)$ is the trace of $A_B(G)$, which evidently is equal to $|B|$. Thus, $(-1)^1 c_1 = |B|$.

(3). $(-1)^2 c_2$ is equal to the sum of determinants of all 2×2 principal submatrices of $_B(G)$, that is

$$\begin{aligned} c_2 &= \sum_{1 \le i \le j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1 \le i \le j \le n} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= \sum_{1 \le i \le j \le n} a_{ii}a_{jj} - \sum_{1 \le i \le j \le n} a_{ij}^2 \\ &= \begin{pmatrix} |B| \\ 2 \end{pmatrix} - m \end{aligned}$$

□

Theorem 3.2. *Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of $A_B(G)$, then*

(i). (i) $\sum_{i=1}^{n} \lambda_i = |B|$.

(ii). (ii) $\sum_{i=1}^{n} \lambda_i^2 = |B| + 2m$.

Proof.

(i). Since the sum of the eigenvalues of $A_B(G)$ is the trace of $A_B(G)$, it follows that

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = |B|.$$

(ii). Similarly the sum of squares of the eigenvalues of $A_B(G)$ is the trace of $(A_B(G))^2$. Then

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + \sum_{i \neq j}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j}^{n} a_{ji}^2 \\ &= |B| + 2m. \end{aligned}$$

□

Theorem 3.3. *Let G be a graph of order n and size m and let $\lambda_1(G)$ be the largest eigenvalue of $A_B(G)$. Then $\lambda_1(G) \geq \frac{2m+\gamma_b}{n}$.*

Proof. Let G be a graph of order n and let λ_1 be the largest minimum boundary eigenvalue of $A_B(G)$. Then $\lambda_1 = \max_{X\neq 0}\{\frac{X^tAX}{X^tX}\}$, where X is any nonzero vector and X^t is its transpose and A is a matrix . If we tack $X = J = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

Then we have $\lambda_1 \geq \frac{J^tA_BJ}{J^tJ} = \frac{2m+\gamma_b}{n}$. □

Theorem 3.4. *Let G be a graph with a minimum boundary dominating set B. If the minimum boundary dominating energy $E_B(G)$ of G is a rational number,then $E_B(G) \equiv \gamma_b(G)\ (mod 2)$.*

Proof. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be minimum dominating eigenvalues of a graph G of which $\lambda_1, \lambda_2, ..., \lambda_r$ are positive and the rest are non-positive, then

$$\begin{aligned}
\sum_{i=1}^{n}|\lambda_i| &= (\lambda_1 + \lambda_2 + ... + \lambda_r) - (\lambda_{r+1} + \lambda_{r+2} + ... + \lambda_n) \\
&= 2(\lambda_1 + \lambda_2 + ... + \lambda_r) - (\lambda_1 + \lambda_2 + ... + \lambda_n) \\
&= 2(\lambda_1 + \lambda_2 + ... + \lambda_r) - |B| = 2q - |B|.
\end{aligned}$$

Hence, by Theorem 2.3 we have $E_B(G) = 2q - |B|$, Where $q = \lambda_1 + \lambda_2 + ... + \lambda_r$, and the proof is completed. □

4. Minimum Boundary Dominating Energy of Some Standard Graphs

In this section, we investigate the exact values of the MBD energy of some standard graphs.

Theorem 4.1. *For the complete graph K_n, $n \geq 2$, $E_B(K_n) = (n-2) + \sqrt{n^2 - 2n + 5}$.*

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, ..., v_n\}$, then $\gamma_b = 1$. Hence the MBD set is $B = \{v_1\}$ and

$$A_B(K_n) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}_{n\times n}$$

Characteristic polynomial is

$$f_n(K_n,\lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda & -1 & \cdots & -1 & -1 \\ -1 & -1 & \lambda & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & \lambda \end{vmatrix}_{n\times n} = (\lambda+1)^{n+1}(\lambda^2 - (n-1)\lambda - 1).$$

The MBD spectrum of K_n will be written as

$$MBD\ Spec(K_n) = \begin{pmatrix} -1 & \frac{(n-1)+\sqrt{n^2-2n+5}}{2} & \frac{(n-1)-\sqrt{n^2-2n+5}}{2} \\ n+1 & 1 & 1 \end{pmatrix}$$

Hence, the MBD energy is $E_B(K_n) = (n-2) + \sqrt{n^2 - 2n + 5}$. □

Theorem 4.2. *For $n \geq 2$, the MBD energy of Star graph $K_{1,n-1}$ is equal to $\sqrt{4n-3}$.*

Proof. Let $K_{1,n-1}$ be a star graph with vertex set $V = \{v_0, v_1, v_2, ..., v_{n-1}\}$, v_0 is the center, then $\gamma_b = 1$. MBD set is $B = \{v_0\}$. Then

$$A_B(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}$$

The characteristic polynomial of $A_B(K_{1,n-1})$ is

$$f_n(K_{1,n-1}, \lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & 0 & \cdots & 0 \\ -1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{n\times n}$$

$$= \lambda^{n-2}(\lambda^2 - \lambda - (n-1)).$$

It follows that the MBD spectrum is

$$MBD\ Spec(K_{1,n-1}) = \begin{pmatrix} 0 & \frac{1+\sqrt{4n-3}}{2} & \frac{1-\sqrt{4n-3}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Therefore, the MBD energy of a star graph is $E_B(K_{1,n-1}) = \sqrt{4n-3}$. □

Theorem 4.3. *For the complete bipartite graph $K_{r,s}$, $r \leq s$, the MBD energy is equal to $(s-2)+\sqrt{s^2-2s+5}+\sqrt{r^2-2r+5}$.*

Proof. For the complete bipartite graph $K_{r,s}$, $(r \leq s)$ with vertex set $V = \{v_1, v_2, ..., v_r, u_1, u_2, ..., u_s\}$, $\gamma_b = 2$, hence the MBD set is $B = \{v_1, u_1\}$. Then

$$A_B(K_{r,s}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 0 \end{pmatrix}_{(r+s)\times(r+s)}$$

The characteristic polynomial of $A_B(K_{r,s})$, where $n = r+s$ is

$$f_n(K_{r,s},\lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ -1 & \lambda & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & \lambda & \cdots & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda-1 & -1 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & \lambda \end{vmatrix}_{(r+s)\times(r+s)} = (\lambda+1)^{s-2}(\lambda^2-(s-1)\lambda-1)(\lambda^2-(r-1)\lambda-1)$$

it follows that

$$MBD\ Spec(K_{r,s}) = \begin{pmatrix} -1 & \frac{(s-1)+\sqrt{s^2-2s+5}}{2} & \frac{(s-1)-\sqrt{s^2-2s+5}}{2} & \frac{(r-1)+\sqrt{r^2-2r+5}}{2} & \frac{(r-1)-\sqrt{r^2-2r+5}}{2} \\ s-2 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence, the MBD energy is

$$E_B(K_{r,s}) = (s-2) + \sqrt{s^2-2s+5} + \sqrt{r^2-2r+5}.$$

□

Definition 4.4. *The double star graph $S_{n,m}$ is the graph constructed from union $K_{1,n-1}$ and $K_{1,m-1}$ by join whose centers v_0 with u_0. Then $V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1}) = \{v_0, v_1, ..., v_{n-1}, u_0, u_1, ..., u_{m-1}\}$ and $E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j; 1 \leq i \leq n-1, 1 \leq j \leq m-1\}$.*

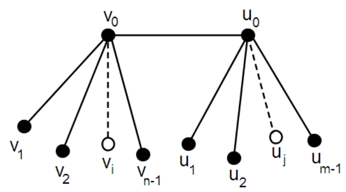

Figure 2. **Double Star Graph $S_{n,m}$**

Theorem 4.5. *For the double star graph $S_{r,r}$ with $r \geq 3$, the MBD energy is equal to $2(\sqrt{r-1}+\sqrt{r})$.*

Proof. For the double star $S_{r,r}$ with $V(S_{r,r}) = \{v_0, v_1, ..., v_{r-1}, u_0, u_1, ..., u_{r-1}\}$, $\gamma_b = 2$, hence the MBD set is $B = \{v_0, u_0\}$. Then

$$A_B(S_{r,r}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}_{2r\times 2r}$$

The characteristic polynomial of $A_B(S_{r,r})$ is

$$f_n(S_{r,r},\lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 & 0 & \cdots & 0 \\ -1 & \lambda & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & \lambda-1 & -1 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & \lambda \end{vmatrix}_{2r\times 2r}$$

$$= \lambda^{2r-4}(\lambda^2-(r-1))(\lambda^2-2\lambda-(r-1)).$$

Then the MBD spectrum of $S_{r,r}$ is

$$MBD\ Spec(S_{r,r}) = \begin{pmatrix} 0 & \sqrt{r-1} & -\sqrt{r-1} & 1+\sqrt{r} & 1-\sqrt{r} \\ 2r-4 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Hence, the MBD energy of $S_{r,r}$ is

$$E_B(S_{r,r}) = 2(\sqrt{r-1}+\sqrt{r}).$$

□

5. Bounds on Minimum Boundary Dominating Energy of Graphs

Theorem 5.1. *Let G be a connected graph of order n and size m. Then*

$$\sqrt{2m+\gamma_b} \leq E_B(G) \leq \sqrt{n(2m+\gamma_b)}.$$

Proof. Consider the Cauchy-Schwartiz inequality

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n (a_i)^2 \right)\left(\sum_{i=1}^n (b_i)^2 \right)$$

By choose $a_i = 1$ and $b_i = |\lambda_i|$ and by Theorem 2.3, we get

$$\begin{aligned}(E_B(G))^2 = \left(\sum_{i=1}^n |\lambda_i| \right)^2 &\leq \left(\sum_{i=1}^n 1 \right)\left(\sum_{i=1}^n (\lambda_i)^2 \right) \\ &\leq n(2m+|B|) \\ &\leq n(2m+\gamma_b(G)).\end{aligned}$$

Therefore, the upper bound is hold. For the lower bound since

$$\left(\sum_{i=1}^n |\lambda_i| \right)^2 \geq \sum_{i=1}^n \lambda_i^2,$$

it follows by Theorem 2.3 that

$$(E_B(G))^2 \geq \sum_{i=1}^n \lambda_i^2 = 2m+|B| = 2m+\gamma_b(G).$$

Therefore, the lower bound is hold. □

Theorem 5.2. *Let G be a connected graph of order n and size m. Then*

$$\sqrt{n+1} \leq E_B(G) \leq n\sqrt{n}.$$

Proof. Since for any graph $\gamma_b(G) \leq n-2$ (see[2]), it follows that by using Theorem 2.9 and well-known result $2m \leq n^2 - n$, we have

$$E_B(G) \leq \sqrt{n(2m+\gamma_b)} \leq \sqrt{n[n^2-n+(n-2)]} \leq n\sqrt{n}.$$

For the lower bound, Since for any connected graph $n \leq 2m$ and $\gamma_b(G) \geq 1$ (see[2]), it follows by Theorem 2.9 that

$$E_B(G) \geq \sqrt{2m+\gamma_b} \geq \sqrt{n+1}.$$

□

Similar to Koolen and Moultons [11], upper bound for $E_B(G)$ is given in the following theorem.

Theorem 5.3. *Let G be a connected graph of order n and size m. Then*

$$E_B(G) \leq \frac{2m+\gamma_b}{n} + \sqrt{(n-1)[2m+\gamma_b - (\frac{2m+\gamma_b}{n})^2]}.$$

Proof. Consider the Cauchy-Schwartiz inequality

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n (a_i)^2 \right) \left(\sum_{i=1}^n (b_i)^2 \right)$$

By choose $a_i = 1$ and $b_i = |\lambda_i|$ and by Theorem 2.3, we get

$$\left(\sum_{i=1}^n |\lambda_i| \right)^2 \leq \left(\sum_{i=1}^n 1 \right) \left(\sum_{i=1}^n (\lambda_i)^2 \right)$$

Hence, by Theorem 2.3 we have

$$(E_B - |\lambda_1|)^2 \leq (n-1)(2m+\gamma_b - \lambda_1^2)$$

Therefore,

$$E_B \leq \lambda_1 + \sqrt{(n-1)(2m+\gamma_b-\lambda_1^2)}.$$

From Theorem 2.4 we have $\lambda_1(G) \geq \frac{2m+\gamma_b}{n}$. Since $f(x) = x + \sqrt{(n-1)(2m+\gamma_b - x^2)}$ is a decreasing function, we have

$$f(\lambda_1) \leq f(\frac{2m+\gamma_b}{n}).$$

Thus,

$$E_B \leq f(\lambda_1) \leq f(\frac{2m+\gamma_b}{n}).$$

Therefore,

$$E_B(G) \leq \frac{2m+\gamma_b}{n} + \sqrt{(n-1)[2m+\gamma_b - (\frac{2m+\gamma_b}{n})^2]}.$$

□

Theorem 5.4. *Let G be a connected graph of order n and size m. If $D = det(A_B(G))$, then*

$$E_B(G) \geq \sqrt{2m + \gamma_b(G) + n(n-1)D^{\frac{n}{2}}}.$$

Proof. Since

$$(E_B(G))^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)\left(\sum_{i=1}^{n}|\lambda_i|\right) = \sum_{i=1}^{n}|\lambda_i|^2 + 2\sum_{i\neq j}|\lambda_i||\lambda_j|.$$

Employing the inequality between the arithmetic and geometric means, we get

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}$$

Thus

$$\begin{aligned}
(E_B(G))^2 &\geq \sum_{i=1}^{n}|\lambda_i|^2 + \frac{1}{n(n-1)}\left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}} \\
&\geq \sum_{i=1}^{n}|\lambda_i|^2 + \frac{1}{n(n-1)}\left(\prod_{i=j}|\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\
&= \sum_{i=1}^{n}|\lambda_i|^2 + \frac{1}{n(n-1)}\left|\prod_{i=j}\lambda_i\right|^{\frac{2}{n}} \\
&= 2m + \gamma_b(G) + n(n-1)D^{\frac{n}{2}}.
\end{aligned}$$

□

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