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Connected Boundary Edge Domination In Graphs

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Abstract: In this paper we define the connected edge dominating set for a connected subgraph. We introduce the connected boundary edge dominating set and analogous to the connected boundary edge domination number, we define the connected boundary edge domatic number in graphs. Exact values of some standard graphs are obtained and some other interesting results are established.
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1-Introduction

By a graph we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [4].

Let $G = (V, E)$ be a graph. A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to at least one vertex in S. An excellent treatment of fundamentals of domination in graphs and several advanced topics in domination are given in Haynes et al [5]. Sampathkumar and Walikar [9] introduced the concept of connected domination. A dominating set S of a connected graph G is called a connected dominating set of G if the induced subgraph his is connected. The minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$.

As an analogy to vertex domination, the concept of edge domination was introduced by Mitchell and Hedetniemi [7]. A set $X \subseteq E$ is said to be an edge dominating set if every edge in $E - X$ is adjacent to some edge in X. The edge domination number of G is the cardinality of a smallest edge dominating set of G and is denoted by γ'. B. ZELINKA [10] introduced the concept of edge domatic number of G. The maximum order of a partition of E into edge dominating sets of G is called the edge domatic number of G and is denoted by d'. The degree of an edge $e = uv$ of G is defined by $deg(e) = deg(u) + deg(v) - 2$. The minimum (maximum) degree of an edge is denoted by $\delta^{'}(\Delta^{'})$. The concept of connected edge domination was introduced by Arumugam.S. and S.Velammal [1]. An edge dominating set X of a connected graph G is called a connected edge dominating set if the edge induced subgraph $\langle X \rangle$ is connected. The connected edge domination number γ_c' of G is the minimum cardinality of a connected edge dominating set of G.

The concept of boundary domination was introduced by KM. Kathiresan, G. Marimuthu and M. Sivanandha Saraswathy [6].

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. For $i \neq j$ a vertex v_i is a boundary vertex of v_j if $d(v_j, v_t) \leq d(v_i, v_j)$ for all $v_t \in N(v_i)$ (see [3]).

A vertex v is called a boundary neighbor of u if v is a nearest boundary of u. If $u \in V$, then the boundary neighbourhood of u denoted by $N_b(u)$ is defined as

$N_b(u) = \{v \in V : d(u, w) \leq d(u, v)$ for all $w \in N(u)\}$. The cardinality of $N_b(u)$ is denoted by $deg_b(u)$ in G. The maximum and minimum boundary degree of a vertex in G are denoted respectively by $\Delta_b(\delta_b)$. That is $\Delta_b(G) = max_{u \in V} |N_b(u)|$ and $\delta_b(G) = min_{u \in V} |N_b(u)|$.

A vertex u boundary dominate a vertex v if v is a boundary neighbor of u. A subset B of $V(G)$ is called a boundary dominating set if every vertex of $V - B$ is boundary dominated by some vertex of B. The minimum taken over all boundary dominating sets of a graph G is called the boundary domination number of G and is denoted by $\gamma_b(G)$.

The distance $d(e_i, e_j)$ between two edges in $E(G)$ is defined as the distance between the corresponding vertices e_i and e_j in the line graph of G, or if $e_i = uv$ and $e_j = u'v'$, the distance between e_i and e_j in G is defined as follows:

$$d(e_i, e_j) = \min\{ d(u, u'),\ d(u, v'),\ d(v, v'),\ d(v, u')\}.$$

Puttaswamy and Mohammed Alatif [8] introduced the concept of boundary edge domination. A subset $S \subseteq E$ is called a boundary edge dominating set if every edge of $E - S$ is boundary edge dominated by some edge of S. The minimum taken over all edge boundary dominating sets of a graph G is called the boundary edge domination number of G and is denoted by $\gamma_b'(G)$. We need the following theorems.

Theorem 1.1. [8]**.**

a. For any path P_n , $n \geq 3$, $\gamma_b^{'}(P_n) = n - 3$.

b. For any complete graph K_n , $n \geq 4$, $\gamma_b^{'}(K_n) = 3$.

c. If G is a connected graph of size $m \geq 3$, then $\gamma_b^{'} \leq m - \Delta_b^{'}$.

Theorem 1.2. [10]**.** For any graph G, $\delta^{'} \leq d^{'} \leq \delta^{'} + 1$.

2. Results

Definition 2.1. A boundary edge dominating set X of a connected graph G is called a connected boundary edge dominating set if the edge induced subgraph $\langle X \rangle$ is connected. The boundary connected edge domination number $\gamma_{cb}^{\prime}$ of G is the minimum cardinality of a connected boundary edge dominating set of G.

We supposed that G, $L(G)$ are connected because if the line graph has more than one component the boundary edge dominating set has at least one edge from every component of $L(G)$ and then $\langle X \rangle$ is not connected, and conversely if $L(G)$ has a minimum connected boundary edge dominating set X and hence connected boundary edge number then $\langle X \rangle$ is connected that means $L(G)$ is connected according to that we state the following observation.

Theorem 2.2. A connected boundary edge dominating set exist for a line graph $L(G)$ if and only if $L(G)$ is connected.

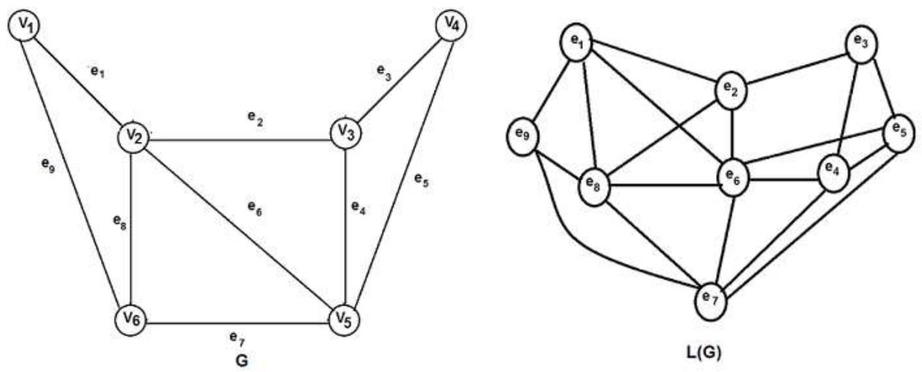

Figure 1: G and L(G)

Example 2.3. Let G be a graph as in the Figure 1, we can find the edge dominating set is $\{e_1, e_4\}$, connected edge dominating set is $\{e_1, e_4, e_6\}$, boundary edge dominating set is $\{e_3, e_9\}$ and connected boundary edge dominating set is $\{e_7, e_8, e_9\}$.

Then $\gamma^{'}(G) = 2$, $\gamma_c^{'}(G) = 3$, $\gamma_b^{'}(G) = 2$ and $\gamma_{cb}^{'}(G) = 3$.

From the definition of line graph and the boundary edge domination the following Proposition is immediate**.**

Observation 2.4. For any graph G, we have, $\gamma'_{cb}(G) = \gamma_{cb}(L(G))$.

By the definition of line graph, we have $V\left(L(G)\right) = E(G)$, and

1. $|V\,[L(P_n)]| = n-1,\ \Delta'_b(P_n) = 2$ and $\delta'_b(P_n) = 1$.
2. $|V\,[L(C_n)]| = n\ and\ \Delta'_b(C_n) = \delta'_b(C_n) = 2$.
3. $|V\,[L(K_{1,n})]| = n\ and\ \Delta'_b(K_{1,n}) = \delta'_b(K_{1,n}) = n-1$.
4. $|V\,[L(K_n)]| = \frac{n(n-1)}{2}\ and\ \Delta'_b(K_n) = \delta'_b(K_n) = \frac{(n-2)(n-3)}{2}$.
5. $|V\,[L(K_{r,s})]| = rs\ and\ \Delta'_b(K_{r,s}) = \delta'_b(K_{r,s}) = r+s-1$.
6. $|V\,[L(B_{r,s})]| = r+s+1\ and\ \Delta'_b(B_{r,s}) = \delta'_b(B_{r,s}) = r+s$.

The connected boundary edge domination number of some standard graphs are given below.

Theorem 2.5.

1. $\gamma'_{cb}(K_n) = 3;\ n \geq 4$.

2. $\gamma'_{cb}(P_n) = \begin{cases} 1 & if\ n = 3,4, \\ 2 & if\ n = 5,6,7, \\ n-4 & if\ n \geq 8\,. \end{cases}$

Theorem 2.6. For a cycle graph C_n , $\gamma'_{cb}(C_n) = \begin{cases} 1 & if\ n = 3, \\ 2 & if\ n = 4,5, \\ n-4 & if\ n \geq 6. \end{cases}$

Proof.

The result is obvious if $n \leq 5$. Suppose $n \geq 6$, since $L(C_n) = C_n,\ d(e_i, e_j) \leq d(e_i, e_t)$ for $all\ e_j \in N(e_i), e_t \in N_b(e_i), and\ \Delta'_b(C_n) = \delta'_b(C_n) = 2$, then $X = \{e_i\,; i = 1,2,\dots,n-4\}$ is a connected boundary edge dominating set of C_n , so that $|X| = \gamma'_{\mathrm{cb}}(\mathrm{C_n}) = n-4\,; n \geq 6$. Hence

$$\gamma'_{cb}(C_n) = \begin{cases} 1 & if\ n = 3, \\ 2 & if\ n = 4,5, \\ n-4 & if\ n \geq 6. \end{cases}$$

□

Theorem 2.7. For a complete bipartite graph $K_{r,s},\ 2 \leq r \leq s,\ \gamma'_{\mathrm{cb}}(K_{r,s}) = r$.

Proof.

Let $G = L(K_{r,s})$and let (V_1, V_2) be the bipartition of $K_{r,s}$ such that $2 \leq r \leq s, |V_1| = r\ and$

$|V_2| = s$. Then for each $v \in V_1$ there exists r bipartition of the sets of all edges incident with v in G, say $X(v) = \{X(v_1), X(v_2), \dots, X(v_r)\}$ such that $X(v_1) = \{ e_i \,; i = 1,2,\dots,r\}$, $X(v_2) = \{ f_j \,; j = 1,2,\dots,r\}$,…, $X(v_r) = \{ g_t \,; t = 1,2,\dots,r\}$ and $e_i, f_j, \dots, g_t$ are both adjacent if $i = j = \cdots = t$. Since $d(e_i, f_j) \leq d(e_i, g_t)$ for $all\ f_j \in N(e_i)$, $g_t \in N_b(e_i)$ $(i = j \neq t)$ and $\Delta_b^{'} = \delta_b^{'} = r + s - 1$ then $X = \{ e_i, f_j, \dots, g_t, i = j = \cdots = t;\ i = 1, 2, \dots, r\}$ is a connected boundary edge dominating set of G, so that $|X| = \gamma'_{\mathrm{cb}}(K_{r,s}) = r$. □

Theorem 2.8. For any graph G, $\gamma'_{\mathrm{cb}}(G) = 1$ if and only if $G \cong K_{1,n}$ or $B_{r,s}$.

Proof.

Suppose that $\gamma'_{\mathrm{cb}}(G) = 1$, Let S denote the set of all connected boundary edge dominating set of G such that $|S| = 1$, we have $\gamma_{cb}^{'}(G) = \gamma_{cb}(L(G))$, then $\gamma_{cb}(L(G)) = 1$. Since $L(K_{1,n}) = K_n$, $n \geq 3$, $\gamma_{cb}(K_n) = \gamma_b(K_n) = 1$ and $L(B_{r,s})$ is the one point say e union of 2 complete graphs K_r and K_s. Therefore $G \cong K_{1,n}$ or $B_{r,s}$.

Conversely, suppose $G \cong K_{1,n}$ or $B_{r,s}$, the line graph of $K_{1,n}$, $n \geq 3$ is K_n and the line graph of the bistar $B_{r,s}$ is the one point say e union of 2 complete graphs K_r and K_s.

Hence $\gamma'_{\mathrm{cb}}(G) = \gamma_{cb}(K_n) = \gamma_b(K_n) = 1$. □

Theorem 2.9. For any connected graph G, $\gamma'_{\mathrm{b}}(G) \leq \gamma'_{\mathrm{cb}}(G)$.

Proof.

From the definition of the connected boundary edge dominating set of a graph G, it is clearly that for any graph G any connected boundary edge dominating set X is also a boundary edge dominating set. Hence $\gamma'_{\mathrm{b}}(G) \leq \gamma'_{\mathrm{cb}}(G)$. □

Proposition 2.10. Let e be an edge of a connected graph G. Then $E - N_{cb}(e)$ is a boundary edge dominating set for G.

Theorem 2.11. If G is a connected graph of size $m \geq 3$, then $\gamma'_{\mathrm{cb}}(G) \leq m - \Delta_{cb}^{'}(G)$

Proof.

Let e be an edge of a connected graph G. Then by the above proposition, $E - N_{cb}(e)$ is a connected boundary edge dominating set for G. $\left(N_{cb}(e) = \Delta_{cb}^{'}(e)\right)$. But $|N_{cb}(e)| \geq 1$. Thus $\gamma'_{\mathrm{cb}}(G) \leq m - 1$, suppose $\gamma'_{\mathrm{cb}}(G) = m - 1$. Then there exists a unique edge e^* in G such that e^* is a boundary edge neighbour of every edge of $E - \{ e^*\}$, this is a contradiction to the fact that in a graph there exist at least two boundary edges. Thus $\gamma'_{\mathrm{cb}}(G) \leq m - 2$. Hence $\gamma'_{\mathrm{cb}}(G) \leq m - \Delta_{cb}^{'}(G)$. □

Corollary 2.12. For any connected graph G of order $n \geq 5$, then $\gamma'_{\mathrm{cb}}(G) \leq n-4$.

Theorem 2.13. [3] If G is a connected graph of order $n \geq 4$, then $\gamma'_{\mathrm{c}}(G) \leq n-2$.

Observation 2.14. If G is a connected graph of order $n \geq 4$, then

$$\gamma'_{\mathrm{b}}(G) \leq \gamma'_{\mathrm{cb}}(G) \leq \gamma'_{\mathrm{c}}(G) \leq n-2.$$

Theorem 2.15. If G is a connected graph of size $m \geq 6$, then $\gamma'_{\mathrm{cb}}(G) \geq \left\lceil \frac{m}{1+\Delta'_{cb}} \right\rceil$.

Proof. We have four cases:

Case 1: If $G \cong P_n$, since $|V[L(P_n)]| = n-1 \; and \; \Delta'_{cb}(P_n) = 2$, then $\left\lceil \frac{m}{1+\Delta'_{cb}} \right\rceil = \left\lceil \frac{n-1}{3} \right\rceil \leq n-4 = \gamma'_{\mathrm{cb}}(P_n)$.

Case 2: If $G \cong C_n$, since $|V[L(C_n)]| = n \; and \; \Delta'_{cb}(C_n) = 2$, then $\left\lceil \frac{m}{1+\Delta'_{cb}} \right\rceil = \left\lceil \frac{n}{3} \right\rceil \leq n-4 = \gamma'_{\mathrm{cb}}(C_n)$.

Case 3: If $G \cong K_n$, since $|V[L(K_n)]| = \frac{n(n-1)}{2} \; and \; \Delta'_{cb}(K_n) = \frac{(n-2)(n-3)}{2}$, then $\left\lceil \frac{m}{1+\Delta'_{cb}} \right\rceil = \left\lceil \frac{\frac{n(n-1)}{2}}{\frac{(n-2)(n-3)}{2}+1} \right\rceil \leq 1 + \left\lceil \frac{1}{(n-2)(n-3)+2} \right\rceil \leq 3 = \gamma'_{\mathrm{cb}}(K_n)$.

Case4: If $G \cong K_{1,n}$, since $\left|V[L(K_{1,n})]\right| = n \; and \; \Delta'_b(K_{1,n}) = n-1$. then $\left\lceil \frac{m}{1+\Delta'_{cb}} \right\rceil = \left\lceil \frac{n}{n} \right\rceil = 1 = \gamma'_{\mathrm{cb}}(K_{1,n})$.

Hence $\gamma'_{cb}(G) \geq \left\lceil \frac{m}{1+\Delta'_{cb}} \right\rceil$. □

From the Theorems 2.11 and 2.15, we obtained the upper and lower bounds of the connected boundary edge domination of the graph G.

Observation 2.16. For any graph G, we have, $\left\lceil \frac{m}{1+\Delta'_{cb}} \right\rceil \leq \gamma'_{\mathrm{cb}}(G) \leq m - \Delta'_{cb}(G)$.

Theorem 2.17. For any (n, m) connected graph G with $\delta'_{cb} \geq 1$, then

$$\left\lceil \frac{m}{1+\Delta'_{cb}} \right\rceil \leq \gamma'_{\mathrm{cb}} \leq 2m-n.$$

Proof.

Let G be any (n, m) connected graph, then by the Observation 2.14 we have $\gamma'_{\mathrm{cb}}(G) \leq n-2$, then $\gamma'_{\mathrm{cb}}(G) \leq n-2 = 2(n-1)-n = m$. And by Theorem 2.15. We get $\left\lceil \frac{m}{1+\Delta'_{cb}} \right\rceil \leq \gamma'_{\mathrm{cb}}(G)$.

Hence $\left\lceil \frac{m}{1+\Delta'_{cb}} \right\rceil \leq \gamma'_{\mathrm{cb}}(G) \leq 2m-n$. □

Theorem 2.18. For any (n, m) connected graph G, $\gamma'_{\mathrm{c}}(G) + \gamma'_{\mathrm{cb}}(G) \leq m+1$.

Proof.

Let $e \in V(L(G))$, then $N_c^{'}(e) \cup N_{cb}^{'}(e) \cup \{e\} = E$, $|N_c^{'}(e)| + |N_{cb}^{'}(e)| + 1 = m$.
and $\Delta_c^{'} + \Delta_{cb}^{'} + 1 = m$. But we have $\gamma_c'(G) \leq m - \Delta_c^{'}$ and $\gamma_{cb}'(G) \leq m - \Delta_{cb}^{'}$.
Therefore $\gamma_c'(G) + \gamma_{cb}'(G) \leq 2m - (\Delta_c^{'} + \Delta_{cb}^{'}) = 2m - m + 1 = m + 1$.

Hence $\gamma_c'(G) + \gamma_{cb}'(G) \leq m + 1$. □

Theorem 2.19. Let G be a graph without any boundary isolated edges and with diameter two, then $\gamma_{cb}'(G) \leq \delta_{cb}' + 1$.

Proof.

Let e be any edge with $deg_b(e) = \delta_{cb}'(G)$. Then obviously $N_{cb}'(e)$ is connected boundary edge dominating set and hence $\gamma_{cb}'(G) \leq \delta_{cb}' + 1$. □

3 Connected Boundary Edge Domatic Number

The maximum order of a partition of the vertex set V of a graph G into dominating sets is called the domatic number of G and is denoted by $d(G)$. For a survey of results on domatic number and their variants we refer to Zelinka [10]. In this section we present a few basic results on the connected boundary edge domatic number of a graph.

Definition 3.1. Let $G = (V, E)$ be a connected graph. The maximum order of a partition of E into connected boundary edge dominating sets of G is called the connected boundary edge domatic number of G and is denoted by $d_{cb}'(G)$.

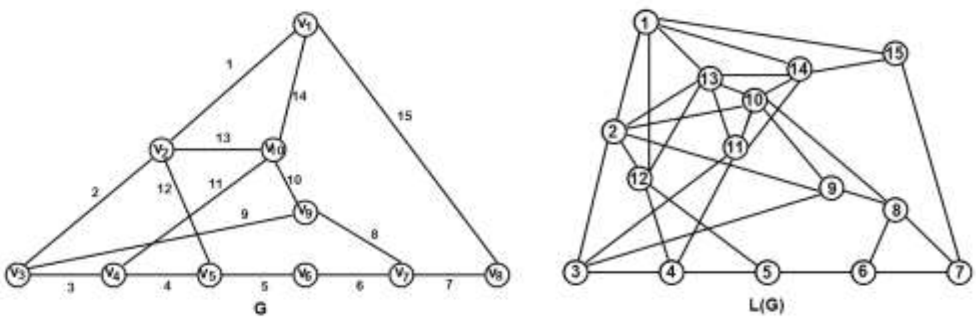

Figure 2: G and L(G)

Example 3.2. In Figure 2,

$\{\{1, 8, 11, 12\}, \{3, 5, 7, 13\}, \{4, 6, 10, 15\}\}$ is an edge domatic partition, $\{2, 8, 14, 12, 10\}$ is a connected edge domatic partition, $\{\{1, 6, 8\}, \{3, 9, 11\}, \{7, 14, 15\}, \{2, 4, 10\}, \{5, 12, 13\}\}$ is a boundary edge domatic partition and $\{\{1,2,9\},\{7,14,15\},\{4,5,12\},\{8,10,13\}\}$ is a connected boundary edge domatic partition.

Then $d^{'} = 3,\ d_c^{'} = 1,\ d_{cb}^{'} = 4$ and $d_b^{'} = 5$. Whereas $\gamma^{'} = 4,\ \gamma_c^{'} = 5,$ and $\gamma_{cb}^{'} = \gamma_b^{'} = 3$.

Observation 3.3. For any graph G, we have

1. $d_b'(G) \le d_{cb}'(G)$.
2. $d_{cb}'(G) \le \left\lfloor \frac{m}{\gamma_{cb}'(G)} \right\rfloor$.

We first determine the connected boundary domatic edge number of some standard graphs. We observe that

Theorem 3.4.

1. $d_{cb}'(C_n) = \begin{cases} 3 & \text{if } n = 6, \\ 2 & \text{if } n = 4,5,7,8, \\ 1 & \text{if } n \ge 9\,. \end{cases}$
2. $d_{cb}'(P_n) = \begin{cases} 2 & if\ n = 4, \\ 1 & if\ n \ge 5. \end{cases}$
3. $d_{cb}'(K_n) = \begin{cases} \left\lceil \frac{n}{4} \right\rceil & if\ n = 2k, \\ \left\lceil \frac{n}{4} \right\rceil & if\ n = 2k+1. \end{cases}$
4. $d_{cb}'(B_{r,s}) = 1$.

Theorem 3.5. For a complete bipartite graph $K_{r,s},\ 2 \le r \le s,\ d_{\mathrm{cb}}^{'}(K_{r,s}) = s$.

Proof.

Let $(V1, V2)$ be the bipartition of $K_{r,s}$ such that $2 \le r \le s, |V_1| = r$ and $|V_2| = s$. From the Theorem 2.7, then For each $u \in V_2$ there exist s bipartition of the sets of all edges incident with u in $K_{r,s}$ say $X(u)$. Since $d(e_i, f_j) \le d(e_i, g_t)$ for $all\ f_j \in N(e_i),\ g_t \in N_b(ei)$, then $\{X(u) \,/\, u \in V_2\}$ forms a connected boundary edge domatic partition of $K_{r,s}$, so that $d_{\mathrm{cb}}^{'}(K_{r,s}) \ge s$. Further $\gamma_{\mathrm{cb}}^{'}(K_{r,s}) = r$ and hence $d_{\mathrm{cb}}^{'}(K_{r,s}) \le \left\lfloor \frac{rs}{\gamma_{\mathrm{cb}}^{'}} \right\rfloor = \frac{rs}{r} = s$. Thus $d_{\mathrm{cb}}^{'}(K_{r,s}) = s$. □

Theorem 3.6. For a star graph $K_{1,n},\ d_{\mathrm{cb}}^{'}(K_{1,n}) = n$.

Proof.

Let $K_{1,n}$ be a star graph with $n \ge 2$ such that its line graph of order n, since $\gamma_{\mathrm{cb}}^{'}(K_{1,n}) = 1$, it follows that $d_{\mathrm{cb}}^{'}(K_{1,n}) \le \left\lfloor \frac{n}{\gamma_{\mathrm{cb}}^{'}} \right\rfloor = n$.

To prove the reverse inequality, since $L(K_{1,n}) = K_n, n \ge 3$ and $\gamma_{\mathrm{cb}}^{'}(K_n) = 1$, then

$\{X(i) = \{e_i\};\ i = 1,2,\dots,n\}$ forms a connected boundary edge domatic partition of $L(K_{1,n})$, where $X(1) = \{e_1\},\ X(2) = \{e_2\},\dots, X(n) = \{e_n\},$ so that $d'_{\mathrm{cb}}(K_{1,n}) \geq n$. Hence $d'_{\mathrm{cb}}(K_{1,n}) = n$. □

Proposition 3.7. For any (n,m) connected graph G ,

1. $1 \leq d'_{\mathrm{cb}}(G) \leq m$.
2. $d'_{\mathrm{cb}}(G) \leq \delta'_{cb}(G) + 1$.

Theorem 3.8. For any (n,m) connected graph G , $d'_{\mathrm{c}}(G) \leq d'_{\mathrm{cb}}(G) \leq d'_{\mathrm{b}}(G)$.

Proof.

Since $d'_{\mathrm{cb}} \leq \left\lfloor\frac{m}{\gamma'_{cb}}\right\rfloor \leq \frac{m}{\gamma'_{cb}}$, $d'_{\mathrm{b}} \leq \left\lfloor\frac{m}{\gamma'_{b}}\right\rfloor \leq \frac{m}{\gamma'_{b}}$ and $d'_{\mathrm{c}} \leq \left\lfloor\frac{m}{\gamma'_{c}}\right\rfloor \leq \frac{m}{\gamma'_{c}}$. From the Theorem 2.9 and the Observation 2.14 we have $\gamma'_b \leq \gamma'_{cb} \leq \gamma'_c$ and $\frac{m}{\gamma'_c} \leq \frac{m}{\gamma'_{cb}} \leq \frac{m}{\gamma'_b}$, then $d'_{\mathrm{c}}(G) \leq d'_{\mathrm{cb}}(G) \leq d'_{\mathrm{b}}(G)$. □

Theorem 3.9. For any (n,m) connected graph G , $d'_{\mathrm{cb}}(G) \geq \left\lfloor\frac{m}{m-\delta'_{cb}(G)}\right\rfloor$.

Proof.

Assume that $d'_{\mathrm{cb}} = l$ and $\{S_1, S_2, \dots, S_n\}$ is a partition of E into l connected boundary edge dominating sets, clearly $|S_i| \geq \gamma'_{cb}$ for $i = 1,2,\dots,l$ and we have $m = \sum_{i=1}^{l}|S_i| \geq l\delta'_{cb}$.

Hence $d'_{\mathrm{cb}}(G) \geq \left\lfloor\frac{m}{m-\delta'_{cb}(G)}\right\rfloor$. □

Theorem 3.10. For any connected graph G, $\gamma'_{cb}(G) + d'_{cb}(G) \leq m + 1$ and equality holds if and only if G is isomorphic to $K_{1,n}$.

Proof.

Since $d'_{\mathrm{cb}}(G) \leq \delta'_{cb}(G) + 1$ and $\gamma'_{\mathrm{cb}}(G) \leq m - \Delta'_{cb}(G)$, we have $d'_{\mathrm{cb}} + \gamma'_{\mathrm{cb}} \leq m - \Delta'_{cb} + \delta'_{cb} + 1 \leq m + 1$. Further $d'_{\mathrm{cb}} + \gamma'_{\mathrm{cb}} = m + 1$ if and only if $\gamma'_{\mathrm{cb}} = m - \Delta'_{cb}$, $d'_{\mathrm{cb}} = \delta'_{cb} + 1$ and $\Delta'_{cb} = \delta'_{cb}$ We claim that $\gamma'_{\mathrm{cb}} = 1$. If $\gamma'_{\mathrm{cb}} \geq 2$, then $d'_{\mathrm{cb}} \leq \frac{m}{2}$. Since $d'_{\mathrm{cb}} + \gamma'_{\mathrm{cb}} = m + 1$, we have $\gamma'_{\mathrm{cb}} \geq \frac{m}{2}$. It follows that $d'_{\mathrm{cb}} = 1$ so that $\gamma'_{\mathrm{cb}} = m$ which is a contradiction. Hence $\gamma'_{\mathrm{cb}} = 1$ and $d'_{\mathrm{cb}} = m$. So that G is isomorphic to $K_{1,n}$. □

Theorem 3.11. For any connected graph G, $d'_{cb}(G) + d'_c(G) \leq 2m$ if and only if $\gamma'_{cb}(G) = \gamma'_c(G) = 1$.

Proof.

Suppose that $\gamma'_{cb}(G) = \gamma'_c(G) = 1$, then $\Delta'_{cb} = \delta'_{cb} = \Delta'_c = \delta'_c = m - 1$ and $d'_{cb} + d'_c \leq \delta'_{cb} + \delta'_c + 2 = 2m - 2 + 2 = 2m$.

Conversely, suppose $d'_{cb}(G) + d'_c(G) \leq 2m$, since $d'_{cb} \leq \left\lfloor\frac{m}{\gamma'_{cb}}\right\rfloor$ and $d'_c \leq \left\lfloor\frac{m}{\gamma'_c}\right\rfloor$, then $d'_{cb} \leq \frac{m}{\gamma'_{cb}}, d'_c \leq$

$\frac{m}{\gamma_c'}$ and $d_{cb}' + d_c' \leq \frac{m}{\gamma_{cb}'} + \frac{m}{\gamma_c'} = \frac{m(\gamma_{cb}'+\gamma_c')}{\gamma_{cb}'\gamma_c'}$, therefore $\frac{m(\gamma_{cb}'+\gamma_c')}{\gamma_{cb}'\gamma_c'} \leq 2m$ and equality holds if and only if $\gamma_{cb}'(G) = \gamma_c^{'}(G) = 1$. □

4 Conclusion

In this paper we computed the exact value of the connected boundary edge domination number and the connected boundary edge domatic number for some standard graphs and some special graphs. Also we found some upper and lower bounds for connected boundary edge domination number and connected boundary edge domatic number of graph.

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