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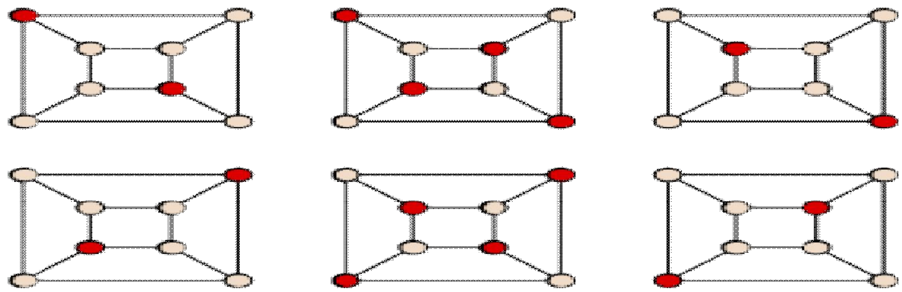

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ON THE INDEPENDENT NEIGHBOURHOOD POLYNOMIAL OF GRAPHS

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ABSTRACT

Graph polynomial is one of the algebraic representations for graph. In this paper we introduce a new type of graph polynomial called an independent neighbourhood polynomial. We obtained the neighbourhood polynomial of some interested standard graphs some characterization by using the independent neighbourhood polynomial of graphs.

KEYWORDS: Graph polynomial, Independent neighbourhood number of a graph AMS Mathematics Subject Classification (2000): 05C69 • 05E15

1. INTRODUCTION

Throughout this paper we will consider only a simple connected graphs, finite and undirected, without loops and multiple edges. In general, we use ⟨X⟩ to denote the subgraph induced by the set of vertices X. N(v) and N[v] denote the open and closed neighbourhood of a vertex v, respectively. For terminology and notations not specifically defined here, we refer reader to [3] and

[5]. For more details about neighbourhood number and its related parameters, we refer to [4] and [6]. A set of vertices in a graph G is a neighbourhood set if $G = \cup_{v \in s} \langle N[v] \rangle$ where $\langle N[v] \rangle$ is the subgraph of G induced by v and all vertices adjacent to v. The neighbourhood number $\eta(G)$ of G is the minimum cardinality of a neighbourhood set. The neighbourhood set s is called an independent neighbourhood set of G if $\langle s \rangle$ is totally disconnected graph and the independent neighbourhood number $n_i(G)$ is the minimum cardinality of an independent neighbourhood set. A neighbourhood set s of G is called connected neighbourhood set if $\langle s \rangle$ is connected. The connected neighbourhood number $n_i(G)$ is the minimum cardinality of a connected neighbourhood set.

The Independent neighbourhood number is not defined for any graph, for example c_5 has no Independent neighbourhood set.

Recently graph polynomials are studied by many authors, domination polynomial are introduced in [1] as :

Let $G = (V, E)$ be any graph with p vertices, Then the domination polynomial of G, $D(G, x) = \sum_{i=\gamma(G)}^{p} d(G, i)x^{ji}$, where $\gamma(G)$ is the domination number of G and $d(G, i)$ is the number of dominating sest in G of size i.

Similarly the neighbourhood polynomial in graph is introduced in [2]. In this paper we introduce the independent neighbourhood polynomial in graphs. Some graphs classification and the independent neighbourhood polynomial of some standard graphs are obtained.

The Independent neighbourhood number is not defined for any graph, for example c_5 has no Independent neighbourhood set. A graph G is called an IN-graph if G has an Independent neighbourhood set, i.e., n_e exist. The neighbourhood polynomial of a graph G has introduced by Anwar Alwardi et al, in [2] . In this paper we study special cases of the neighbourhood polynomial in graph.

2. THE INDEPENDENT NEIGHBOURHOOD POLYNOMIAL IN GRAPH

Definition 2.1. *Let $G = (V, E)$ be an IN-graph with p vertices, Then the Independent neighbourhood polynomial of G, $N_i(G, x) = \sum_{j=\eta_i}^{p} \eta_i(G, j)x^j$, where η_i is the independence neighbourhood number of G and $\eta_i(G, j)$ is the number of Independent neighbourhood set in G of size j.*

The roots of $N_i(G, x)$ are called the independence neighbourhood roots of G and denoted by $Z(N_i(G, x))$.

Example 2.2. Let G be a graph as in Figure 1.

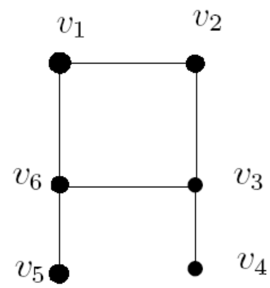

Figure 1.

From Figure 1, it is easy to see that $\eta_i(G) = \eta(G) = 3$ and there are four neighbourhood sets of size 3 which are $\{v_2, v_4, v_6\}, \{v_{1\,,}v_{3\,,}v_5\}, \{v_1, v_3, v_6\} and\ \{v_2, v_3, v_6\}$.similarly, there are five neighbourhood sets $\{v_1, v_2, v_{3,}v_6\}, \{v_{2\,,}v_{3\,,}v_4, v_2\}, \{v_1, v_3, v_4, v_6\}, \{v_1, v_3, v_{5,}v_6\}\ and\{v_2, v_3, v_{5,}v_6\}$. Also there are six neighbourhood sets of size five

$\{v_{2,}v_3, v_4, v_5, v_6\}, \{v_1, v_3, v_4, v_5, v_6\}, \{v_1, v_2, v_4, v_5, v_6\}, \{v_1, v_2, v_3, v_5, v_6\}, \{v_1, v_2, v_3, v_4, v_6\}\ and$

$\{v_1, v_2, v_3, v_4, v_5\}$ and there is only one neighbourhood set of size six.

Therefore $N(G, x) = x^6 + 6x^5 + 5x^4 + 4x^3$. For the independent neighbourhood there are only two independent neighbourhood sets of size three and all the other neighbourhood sets are not independent.

Hence , $N_i(G, x) = 2x^3$.

Observation 2.3. *For any IN-graph G, we have*

$(i)\ N_i(G, x) < N(G, x)$.

$(i)\ degree\ of\ \ N_i(G, x)\ less\ than\ the\ degree\ of\ N(G, x)$.

Proposition 2.4. *(i) For any complete graph* $k_p, N_i(k_p, x) = px$.

(ii) For any totally disconnected graph $\overline{k_p}\ N_i(\overline{k_p}, x) = x^p$

Proof: (i) Let G be a complete graph with p vertices. Then any vertex in G will be neighbourhood set and any other neighbourhood set of more than one vertex will be not independent. Hence there are only p neighbourhood sets all of size one. Hence $N_i(k_p, x) = px$.

(ii) If $G \cong \overline{k_p}$ then clearly there is only independent neighbourhood set of size p.

Hence $N_i(\overline{k_p}, x) = x^p$.

The part (ii) in proposition 2.4 can be generalized as the following proposition.

Proposition 2.5. *Let G be IN-graph .Then $N_i(G,x) = x^p$ if and only if $G \cong \overline{k_p}$.*

Proof: The proof of one side by proposition 2.4 (ii) and the conversely is obvious.

Proposition 2.6. *For any cycle C_{2k} for some $k \geq 2$, $N_i(C_{2k},x) = 2x^k$.*

Proof: Let C_{2k} be labeling as$\{v_1, v_2, v_3, \dots \dots, v_{2k}$, Then clearly $\eta_i(C_{2k}) = k$ and there are only two independent neighbourhood sets of size $\{v_1, v_3, \dots \dots, v_{2k-1}\}$ and $\{v_1, v_3, \dots \dots, v_{2k}\}$ and there is no independent neighbourhood set size more than k. Hence $N_i(C_{2k},x) = 2x^k$

Proposition 2.7. *Let $G \cong W_{2k+1}$ be wheel of $2k+1$ vertices, where $k \geq 2$. Then $N_i(G,x) = x + 2x^k$.*

Proof: Let $G \cong W_{2k+1}$ be wheel with $2k+1$ vertices and labeling as $v, v_1, v_2, v_3, \dots \dots, v_{2k}$ where v is the center of the wheel. Clearly $\eta_i(G) = 1$ the center v is the minimum independent neighbourhood set of G and there are only another two neighbourhood sets of size k which they are $\{v_1, v_3, \dots \dots, v_{2k-1}\}$*and* $\{v_2, v_4, \dots \dots, v_{2k}\}$. Therefore $N_i(G,x) = x + 2x^k$.

Proposition 2.8. *For any complete bipartite graph $k_{m,n}$, we have $N_i(k_{m,n},x) = x^m + x^n$.*

Proof: Obviously $\eta_i(k_{m,n}) = \min\{m,n\}$ and it is easy to see that there are only two independent neighbourhood sets for $k_{m,n}$ which they are the two partite sets of size m and n .

Hence, $N_i(k_{m,n},x) = x^m + x^n$.

Proposition 2.9. *Let G be IN-graph with $n+1$ vertices, Then $N_i(G,x) = x(1 + x^{n-1})$ if and only if $G \cong k_{1,n}$.*

Proof: If $\cong k_{1,n}$, then by proposition 2.8 we get $N_i(G,x) = x(1 + x^{n-1})$.

Conversely, suppose that G has $n+1$ vertices and $N_i(G,x) = x(1 + x^{n-1})$, then clearly G has one neighbourhood of set of size one i.e., there exist a vertex $v \in V(G)$ such that $deg(v) = n$ and there is independent neighbourhood set of size n say s and s does not contain the vertex v which is of full degree because s must be independent. To prove that $G \cong k_{1,n}$ we have to prove that all the vertices other than v are of degree one .Now let there is a vertex $u \neq v$ with degree two, then the set $V - \{v\}$ is not independent set which is contradiction to the fact $s = V - \{v\}$ is independent neighbourhood set . Hence v is of degree n and all the other vertices of degree one .Hence $G \cong k_{1,n}$

Proposition 2.10. *Let $G \cong B(m,n)$ as in Figure 2. Then $N_i(G,x) = x^{m+1} + x^{n+1} + x^{m+n}$ where $m,n \geq 2$.*

Proof: From Figure 2. We get $\eta_i(G) = \min\{m,n\} + 1$. Also there are only three independent neighbourhood sets of$\{u_1, u_2, \dots \dots u_n, v\}\{v_1, v_2, \dots \dots, v_m, u\}$ *and* $\{u_1, \dots \dots, u_n, v_1, \dots \dots, v_m\}$

Figure 2.

Therefore, $N_i(G,x) = x^{m+1} + x^{n+1} + x^{m+n}$.

Proposition 2.11. *Let G be the friendship graph F_m on $2m+1$ vertices .Then $N_i(G,x) = x + 2^m x^m$, where $m \geq 2$.*

Proof: The common vertex of the triangles in the friendship graph from an independent neighbourhood set of size one and there are independent neighbourhood sets of size m can be constructed by selecting one vertex from each triangle other than the center. Therefore, there are 2^m independent neighbourhood sets of cardinality m and no other neighbourhood set of size more than m, because $B(G) = m$. Hence, $N_i(G,x) = x + 2^m x^m$.

Theorem 2.12. *Let G_1 and G_2 be any two graphs with independent neighbourhood polynomials $N_i(G_1\,,x)$and $N_i(G_2\,,x)$ respectively. Let $G \cong G_1 + G_2$.Then $N_i(G,x) = N_i(G_1\,,x) + N_i(G_2\,,x)$.*

Proof: By the definition of joining between two graphs G_1 and G_2, any independent neighbourhood set of G_1 will be independent neighbourhood set of G, similarly any independent neighbourhood set of G_2 will be independent neighbourhood set of G and does not exist any independent neighbourhood set contains vertices from G_1 and another vertices from G_2.

Therefore , $n_i(G,x) = n_i(G_1\,,x) + n_i(G_2\,,x)$.

Hence, $N_i(G,x) = N_i(G_1\,,x) + N_i(G_2\,,x)$.

Lemma 2.13. *Let G be an IN-graph. Then $\eta(G) \leq \eta_i(G) \leq \beta(G)$.*

Proof: Let $S \subseteq V(G)$ be minimum independent neighbourhood set in G. Since any independent neighbourhood set is neighbourhood set. Therefore $\eta(G) \leq |S| = \eta_i(G)$.

To prove that $\eta_i(G) \leq \beta(G)$, we can prove that by contradiction. Suppose that G has minimum independent neighbourhood set D such that$|S| = \eta_i$. Now let $\eta_i(G) > \beta(G)$.Then, since S is independent set that will give us contradiction that there are another independent set S and $|S| > \beta(G)$. Hence $\eta_i(G) \leq \beta(G)$.

Theorem 2.14. *Let G be any IN-graph. Then $N_i(G,x) = kx^t$ for some positive integers k and t if and only if $B(G) = \eta_i(G) = t$.*

Proof: Let G be IN-graph with independent neighbourhood polynomial $N_i(G,x) = kx^t$ for some positive integers k and t then clearly there is only k independent neighbourhood set of G and by Lemma 2.13 $\eta_i(G) \leq B(G)$ that means $\eta_i(G) = B(G) = t$.

Conversely if G is IN-graph and we have $B(G) = \eta_i(G) = t$ then there is only independent sets of size t and no independent sets of size more than t then there is only k independent neighbourhood sets of size t in G.

Hence $N_i(G,x) = kx^t$.

Lemma 2.15. *For any IN-graphs* $G_1, G_2, G_3, \ldots\ldots, G_r$ *then* $\bigcup_{i=1}^{r} G_i$ *is also IN-graph.*

Proof: Let $G \cong G_1 \cup G_2$ and we have G_1 *and* G_2 are IN-graphs and let S_1 and S_2 be the minimum independent neighbourhood sets of G_1 *and* G_2 respectively. It is easy to see that $S_1 \cup S_2$ is the minimum independent neighbourhood set of G. Therefore $G \cong G_1 \cup G_2$ is IN-graph. Suppose that $\bigcup_{i=1}^{k} G_i$ is IN-graph if $G_1, G_2, G_3, \ldots\ldots, G_k$ are In-graphs, we want to prove that $\bigcup_{i=1}^{k+1} G_i$ is also IN-graph. $\bigcup_{i=1}^{k+1} G_i = \bigcup_{i=1}^{k} G_i \cup G_{k+1}$ and if S_{k+1} is the minimum independent neighbourhood set and $\bigcup_{i=1}^{k} S_i$ is the minimum independent neighbourhood set of $\bigcup_{i=1}^{k+1} G_i$ *is* $\bigcup_{i=1}^{k+1} S_i$. Hence, $\bigcup_{i=1}^{k+1} G_i$ is IN-graph.

Hence for any $G_1, G_2, G_3, \ldots\ldots, G_r$, which they are IN-graph $\bigcup_{i=1}^{r} G_i$ is IN-graph.

Proposition. 2.16. Let $G = G_1 \cup G_2$, where G_1, G_2 be any two IN-graphs. Then $(G,x) = N_i(G_1,x)N_i(G_2,x)$.

Proof: Let G_1, G_2 be any two IN-graphs. Then by Lemma 2.15, G is IN-graph it is easy to observe that **Any independent neighbourhood set of size k in G is arising by selecting independent neighbourhood set of size** i **in**G_1**,, and the vertices of independent neighbourhood set of size** $k-j$ **in** G_2**,. And the number of ways of selecting this vertices is equal to the coefficient of the term** x^k **in the polynomial** $N_i(G_1,x)N_i(G_2,x)$**. Hence**$(G,x) = N_i(G_1,x)N_i(G_2,x)$

By mathematical induction it is easy to prove the following Theorem.

Theorem. 2.17. *For any IN-graphs,* $G_1, G_2, G_3, \ldots\ldots, G_r$, *we have* $N_i(G,x) = \prod_{j=1}^{r} N_i(G_j,x)$.

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